Estimates for Solutions of Elliptic Partial Differential Equations with Explicit Constants and Aspects of the Finite Element Method for Second-Order Equations

by Andrew William Cameron

ESTIMATES FOR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH EXPLICIT CONSTANTS AND ASPECTS OF THE FINITE ELEMENT METHOD FOR SECOND-ORDER EQUATIONS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Andrew William Cameron

August 2010

ESTIMATES FOR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH EXPLICIT CONSTANTS AND ASPECTS OF THE FINITE ELEMENT METHOD FOR SECOND-ORDER EQUATIONS

Andrew William Cameron, Ph.D.

Cornell University 2010

The classic L_p-based estimates for solutions of elliptic partial differential equations satisfying general boundary conditions were obtained by Agmon, Douglis, and Nirenberg in 1959. In Chapter 2, we rework these estimates to make their dependence on p explicit. It has long been believed that p enters these estimates as a single multiplicative factor of $(p-1)^{-1}$ for p close to 1 and p for p large. This is verified for second-order equations with boundary conditions of order at most one. Poorer results are obtained for more general problems. Local estimates for solutions of homogeneous equations satisfying homogeneous boundary conditions are also established. These are shown to be independent of p.

Now consider the finite element approximation of a solution of a second-order elliptic partial differential equation. A typical finite element space that we consider is the Lagrange space of continuous functions which are piecewise polynomials on the elements of an unstructured but quasiuniform triangulation of the domain.

As proved by Schatz in 1998, the finite element error is localised in the sense that its L_∞ and W^1_∞ norms in a region depend most strongly on the behaviour of the true solution at points closest to that region. In Chapter 3, we show that the pattern in the positive norm error estimates continues into the L_∞-based negative norms. In particular, the error is localised in the negative norms in the same sense that it is in the positive norms.

A class of a posteriori W^1_∞ estimators for the finite element error was investigated by Hoffman, Schatz, Wahlbin, and Wittum in 2001 for the homogeneous Neumann problem. In Chapter 4, we obtain analogous results for an analogous class of L_∞ estimators. Conditions are given under which these are asymptotically equivalent and asymptotically exact. One specific concrete example is provided.

In the finite element approximation for the homogeneous Dirichlet problem, the computational domain does not typically match the domain on which the original problem is posed. In Chapter 5, we investigate this issue in conjunction with numerical integration. We find that superparametric elements preserve the 1998 weighted L_∞ and W^1_∞ error estimates of Schatz.

BIOGRAPHICAL SKETCH

Andrew William Cameron was born March 31, 1981 in Montreal, Quebec to Alexander Walker Cameron and Mary Kay Emery. His family moved to Rochester, New York, where he attended Allen Creek Elementary School, Pittsford Middle School, and Pittsford Sutherland High School, graduating in 1999. He studied piano at the Eastman School of Music and earned a Diploma in Piano from Eastman's Community Education Division in 1999.

Andrew got a head start on his undergraduate studies by taking courses at the University of Rochester in Summer 1999. He then enrolled at the University of Virginia in Charlottesville, Virginia in Fall 1999. He graduated in Spring 2002, after three years, with a B.S. with Highest Distinction in Physics and Mathematics.

In Fall 2002, Andrew entered the Ph.D. programme in Mathematics at Cornell University in Ithaca, New York. He earned an M.S. in Mathematics in Spring 2006 and completed the requirements for the M.S. in Computer Science in Spring 2009. He graduated in Summer 2010, after eight years at Cornell.

ACKNOWLEDGEMENTS

I would like to thank the members of my committee for helping me in my graduate studies.

My advisor, Alfred H. Schatz, suggested the topics of this thesis and devoted a substantial amount of his time over the past six years to discussing these problems with me. His NSF grant DMS-0612599 helped support my studies.

The idea of pursuing research in partial differential equations first occurred to me after taking a particularly enjoyable course taught by Lars B. Wahlbin in Fall 2003. His notes from a seminar in Spring 1998 were crucial in preparing me to do research on the finite element method.

My understanding of numerical analysis has been further enriched by Stephen A. Vavasis and Charles F. Van Loan. I have had the pleasure of taking two courses taught by each of them in my time at Cornell.

TABLE OF CONTENTS

CHAPTER 1

PRELIMINARIES

1.1 Notation

1.1.1 Integer Sets

For integers i, j, we use the MATLAB-inspired notation $i : j$ to denote the set of integers k with $i \leq k \leq j$.

1.1.2 Points and Sets in $\mathbb{R}^N$

Let $e_1, \ldots, e_N$ denote the standard basis for $\mathbb{R}^N$.

For $x \in \mathbb{R}^N$ and $1 \leq p \leq \infty$, let $|x|_p$ be the Lebesgue norm of x with exponent p and let $|x| = |x|_2$ be the Euclidean norm.

For $x \in \mathbb{R}^N$ and $d > 0$, define the open ball of radius d centred at x

$$B_d(x) = \{y \in \mathbb{R}^N : |y - x| < d\} \tag{1.1}$$

and the open cube of side length $2d$ centred at x

$$C_d(x) = \{y \in \mathbb{R}^N : |x_i - y_i| < d \text{ for all } i \in 1 : N\}. \tag{1.2}$$

Define the open unit ball

$$B^N = \{x \in \mathbb{R}^N : |x| < 1\}, \tag{1.3}$$

the unit sphere

$$\Sigma^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}, \tag{1.4}$$

and the closed unit simplex

$$T^N = \{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in 1 : N \text{ and } \sum_{i=1}^{N} x_i \leq 1\}. \tag{1.5}$$

Define the upper half-space

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}, \tag{1.6}$$

the upper half unit ball $B^N_+ = B^N \cap \mathbb{R}^N_+$, and the upper half unit sphere $\Sigma^{N-1}_+ = \Sigma^{N-1} \cap \mathbb{R}^N_+$.

For $x \in \mathbb{R}^N$, let $x^* = (x_1, \ldots, x_{N-1}, -x_N)$ be the reflection of x in the Nth coordinate. For $V \subset \mathbb{R}^N$, let $V^* = \{x^* : x \in V\}$ be the reflection of V in the Nth coordinate.

Suppose that $U \subset \mathbb{R}^N$. Let $\bar{U}$ denote the closure of U. Let ∂U denote the boundary of U, and, for $x \in \partial\Omega$, let $\nu_U(x)$ denote the outward-pointing unit normal vector to ∂U at x. Define $\operatorname{diam}(U)$, the diameter of U, as twice the radius of the smallest ball that contains all the points of U. We say that U is star-shaped with respect to a point $x \in \mathbb{R}^N$ if $tx + (1-t)y \in U$ for all $y \in U$ and $t \in [0,1]$. We say that U is star-shaped with respect to $V \subset \mathbb{R}^N$ if U is star-shaped with respect to each point in V.

For $U \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, define the point-to-set distance

$$\operatorname{dist}(x, U) = \inf_{y \in U} |x - y|. \tag{1.7}$$

For $U, V \subset \mathbb{R}^N$, define the set-to-set distance

$$\operatorname{dist}(U, V) = \inf_{\substack{x \in U \\ y \in V}} |x - y|. \tag{1.8}$$

For $U \subset \mathbb{R}^N$ and $u : U \to \mathbb{R}$, define the support of u by

$$\operatorname{supp}(u) = \overline{\{x \in U : u(x) \neq 0\}}. \tag{1.9}$$

1.1.3 Multiindices

For $N \geq 1$ an integer, a multiindex of length N is an element of $\{0, 1, 2, \ldots\}^N$. For α a multiindex of length N, define

$$|\alpha| = \sum_{i=1}^{N} \alpha_i \tag{1.10}$$

and

$$\alpha! = \prod_{i=1}^{N} \alpha_i!, \tag{1.11}$$

and, if $x \in \mathbb{R}^N$, define

$$x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}. \tag{1.12}$$

1.1.4 Matrices

For integers $M, N \geq 1$, let $\mathbb{R}^{M\times N}$ be the set of $M \times N$ matrices with real entries. For $A \in \mathbb{R}^{M\times N}$, let A^T denote the transpose of A, and, for $A \in \mathbb{R}^{N\times N}$, let $\det A$ denote the determinant of A. The identity matrix $I \in \mathbb{R}^{N\times N}$ has entries given by the Kronecker delta, $I_{i,j} = \delta_{i,j}$. Elements of $\mathbb{R}^N$ will be identified with elements of $\mathbb{R}^{N\times 1}$. We use the FORTRAN-inspired notation $\mathbb{R}^{(i:j)\times(k:\ell)}$ to denote a matrix whose rows are indexed from i to j and whose columns are indexed from k to ℓ. Similarly, $\mathbb{R}^{i:j}$ will denote a vector whose entries are indexed from i to j.

1.1.5 Differentiation

For U an open subset of $\mathbb{R}^N$, $u : U \to \mathbb{R}$, $i \in 1 : N$, and $x \in U$, let $D_i u(x)$ denote the derivative of u with respect to its ith argument at x. For U an open subset of $\mathbb{R}^N$, $u : U \to \mathbb{R}$, α a multiindex of length N, and $x \in U$, define

$$D^\alpha u(x) = D_1^{\alpha_1} \cdots D_N^{\alpha_N} u(x). \tag{1.13}$$

For U an open subset of $\mathbb{R}^N$, $\Phi : U \to \mathbb{R}^M$, and $x \in U$, define the total derivative $D\Phi(x) \in \mathbb{R}^{M\times N}$ of Φ at x by

$$(D\Phi(x))_{i,j} = D_j\Phi_i(x). \tag{1.14}$$

For $i \geq 0$ an integer and U an open subset of $\mathbb{R}^N$, let $C^i(U)$ denote the set of $u : U \to \mathbb{R}$ for which all derivatives of order at most i exist and are continuous and let $C^i_0(U)$ denote the set of functions in $C^i(U)$ whose supports are bounded subsets of U. Let $C^\infty(U)$ denote the set of $u : U \to \mathbb{R}$ for which all derivatives of all orders exist and are continuous and let $C^\infty_0(U)$ denote the set of functions in $C^\infty(U)$ whose supports are bounded subsets of U.

1.1.6 Integration

For $U \subset \mathbb{R}^N$, let $\mathrm{meas}_N(U)$ denote the N-dimensional Lebesgue measure of U. For U an $(N-1)$-dimensional manifold in $\mathbb{R}^N$, let $\mathrm{meas}_{N-1}(U)$ denote the surface measure of U inherited from the $(N-1)$-dimensional Lebesgue measure on $\mathbb{R}^{N-1}$.

For U an open subset of $\mathbb{R}^N$ and $u : U \to \mathbb{R}$, let $\int_U u$ denote the integral of u over U with respect to the N-dimensional Lebesgue measure on $\mathbb{R}^N$. For U an $(N-1)$-dimensional manifold in $\mathbb{R}^N$ and $u : U \to \mathbb{R}$, let $\int_U u\, dS$ denote the integral of u over U with respect to the surface measure inherited from the $(N-1)$-dimensional Lebesgue measure on $\mathbb{R}^{N-1}$.

1.1.7 Polynomials

For $U \subset \mathbb{R}^N$ and $r \geq 0$ an integer, let $\Pi^r(U)$ denote the set of polynomials in N variables of total degree at most r defined on U. For U an open subset of $\mathbb{R}^N$, $r \geq 0$ an integer, $x \in U$, and $u : U \to \mathbb{R}$, let $T^r_x u \in \Pi^r(\mathbb{R}^N)$ denote the rth-order

Taylor polynomial of u centred at x, defined by

$$T_x^r u(y) = \sum_{|\alpha| \le r} \frac{1}{\alpha!} D^\alpha u(x)(y-x)^\alpha. \tag{1.15}$$

1.1.8 Lebesgue and Sobolev Spaces

Let U be an open subset of $\mathbb{R}^N$. For $1 \le p \le \infty$ and $u : U \to \mathbb{R}$, let $\|u\|_{L_p(U)}$ denote the Lebesgue norm of u with exponent p on U. That is,

$$\|u\|_{L_p(U)} = \begin{cases} \Big(\displaystyle\int_U |u|^p\Big)^{1/p}, & \text{if } 1 \le p < \infty \\ \operatorname*{ess\,sup}_U |u|, & \text{if } p = \infty. \end{cases} \tag{1.16}$$

For $1 \le p \le \infty$ and $u : U \to \mathbb{R}$, let $\|u\|_{W_p^k(U)}$ and $|u|_{W_p^k(U)}$ denote, respectively, the Sobolev norm and seminorm of u with exponent p and differentiability order k on U. That is,

$$\|u\|_{W_p^k(U)} = \begin{cases} \Big(\displaystyle\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p\Big)^{1/p}, & \text{if } 1 \le p < \infty \\ \displaystyle\max_{|\alpha| \le k} \operatorname*{ess\,sup}_U |D^\alpha u|, & \text{if } p = \infty \end{cases} \tag{1.17}$$

and

$$|u|_{W_p^k(U)} = \begin{cases} \Big(\displaystyle\sum_{|\alpha| = k} \int_U |D^\alpha u|^p\Big)^{1/p}, & \text{if } 1 \le p < \infty \\ \displaystyle\max_{|\alpha| = k} \operatorname*{ess\,sup}_U |D^\alpha u|, & \text{if } p = \infty. \end{cases} \tag{1.18}$$

For $1 \le p \le \infty$, let p' denote the Hölder conjugate exponent of p. For $1 < p < \infty$, let $C_p = \max\{(p-1)^{-1}, p\}$. Notice that $C_p \to \infty$ as $p \to 1$ and $p \to \infty$.

For $1 \le p \le \infty$, $k \ge 1$ an integer, and $u : U \to \mathbb{R}$, define the negative Sobolev norm of u with dual exponent p' and dual differentiability order k on U by

$$\|u\|_{W_p^{-k}(U)} = \sup_{\substack{v \in C_0^\infty(U) \\ \|v\|_{W_{p'}^k(U)} = 1}} \Big|\int_U uv\Big|. \tag{1.19}$$

Although the $p = \infty$ case will be of principal interest in this work, it is not standard to include the $p = 1$ or $p = \infty$ cases in the definition of a negative norm. For instance, these cases are explicitly omitted in the treatment of negative norms in the standard references [1, Section 3.13] on Sobolev spaces, [13, Section 1.3.1] on partial differential equations, and [28, Section 2.3.1] on interpolation spaces. These cases appear in several papers on finite element analysis, including [23, Section 1], [25, Section 1], [19, Section 1], and [9, Section 5].

A more general negative norm is sometimes more natural than the negative norm above. For V an open subset of $\mathbb{R}^N$, $1 \le p \le \infty$, $k \ge 1$ an integer, and $u : U \cap V \to \mathbb{R}$, define the negative Sobolev norm of u with dual exponent p' and dual differentiability order k on $U \cap V$, disregarding where $U \cap V$ abuts on ∂V, by

$$\|u\|_{W_p^{-k}(U,V)} = \sup_{\substack{v \in C_0^\infty(U) \\ \|v\|_{W_{p'}^k(U)} = 1}} \left| \int_{U \cap V} uv \right|. \tag{1.20}$$

1.1.9 Product and Dual Spaces

Suppose that U is an open subset of $\mathbb{R}^N$, S is a vector space of functions $u : U \to \mathbb{R}$ with seminorm $|u|_S$, and $\Phi : U \to \mathbb{R}^M$. Define the product seminorm of Φ with respect to $|\cdot|_S$ by

$$|\Phi|_{S^M} = \sum_{i=1}^{M} |\Phi_i|_S. \tag{1.21}$$

If $|\cdot|_S$ is actually a norm, then so is $|\cdot|_{S^M}$.

Suppose that U is an open subset of $\mathbb{R}^N$, S is a vector space of functions $u : U \to \mathbb{R}$ with norm $\|u\|_S$, and $F : S \to \mathbb{R}$. Define the dual norm of F with respect to $\|\cdot\|_S$ by

$$\|F\|_{S'} = \sup_{\substack{u \in S \\ \|u\|_S = 1}} |F(u)|. \tag{1.22}$$

1.1.10 Convolution

Let V, W be open subsets of $\mathbb{R}^N$, $U = \{x - y, x \in W, y \in V\}$, $u : U \to \mathbb{R}$, and $v : V \to \mathbb{R}$. Then we can define $u * v : W \to \mathbb{R}$, the convolution of u and v, by

$$(u * v)(x) = \int_V u(x - y)v(y)\, dy. \tag{1.23}$$

If $0 \in V$ and u is not integrable at 0, then this integral will usually fail to converge. To overcome this issue, we define the principal-value convolution $u \hat{*} v$ of u and v by

$$(u \hat{*} v)(x) = \lim_{\epsilon \to 0^+} \int_{\{y \in V : |x-y| > \epsilon\}} u(x - y)v(y)\, dy. \tag{1.24}$$

1.1.11 Logarithmic Factors

For $0 < a, b \leq 1$ and P a proposition, define the logarithmic factors

$$\ell_a = 1 + \log \frac{1}{a}, \tag{1.25}$$

$$\ell_{P,a} = \begin{cases} \ell_a, & \text{if } P \text{ is true} \\ 1, & \text{if } P \text{ is false,} \end{cases} \tag{1.26}$$

and

$$\ell_{P,a,b} = \begin{cases} \ell_a, & \text{if } P \text{ is true} \\ \ell_b, & \text{if } P \text{ is false.} \end{cases} \tag{1.27}$$

These definitions were inspired by [9, Section 2.3].

1.1.12 Weight Functions

For $U \subset \mathbb{R}^N$ and $w > 0$ a weight parameter, define the weight function $\sigma_{U,w} : \mathbb{R}^N \to \mathbb{R}$ by

$$\sigma_{U,w}(x) = \frac{w}{w + \operatorname{dist}(x, U)}. \tag{1.28}$$

This definition loosely follows [9, Section 2.3]. The weight function defined in [19, Equation 0.7], [20, Equation 1.6], and [22, Equation 1.6] allows only those sets U consisting of a single point in $\mathbb{R}^N$. Observe that $0 < \sigma_{U,w}(x) \leq 1$ for all $x \in \mathbb{R}^N$ and that $\sigma_{U,w}(x)$ gets smaller as x gets farther from U. Furthermore, $w \mapsto \sigma_{U,w}(x)$ is increasing, $t \mapsto \sigma_{U,w}^t(x)$ is decreasing, and $\sigma_{U,w}(x)$ increases as U expands. In [19, Equation 2.10], the multiplicative property

$$\sigma_{\{x\},w}(y)\sigma_{\{y\},w}(z) \leq 2\sigma_{\{x\},w}(z) \tag{1.29}$$

is shown. The generalisation of this is essentially given in [9, Section 2.3],

$$\sigma_{U,w}(y)\sigma_{\{y\},w}(z) \leq 2\sigma_{U,w}(z). \tag{1.30}$$

1.1.13 Weighted Norms

The definitions in this section follow [9, Section 2.3]. For $1 \leq p \leq \infty$, U an open subset of $\mathbb{R}^N$, $V \subset \mathbb{R}^N$, $w > 0$, $t \in \mathbb{R}$ a weight power, and $u \in L_p(U)$, define the weighted norm

$$\|u\|_{L_p(U),V,w,t} = \|\sigma_{V,w}^t u\|_{L_p(U)}. \tag{1.31}$$

For $1 \leq p \leq \infty$, $k \geq 0$ an integer, U an open subset of $\mathbb{R}^N$, $V \subset \mathbb{R}^N$, $w > 0$, $t \in \mathbb{R}$, and $u \in W_p^k(U)$, define the weighted norms and seminorms

$$\|u\|_{W_p^k(U),V,w,t} = \begin{cases} \Big(\displaystyle\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(U),V,w,t} \Big)^{1/p}, & \text{if } 1 \leq p < \infty \\ \displaystyle\max_{|\alpha| \leq k} \|D^\alpha u\|_{L_\infty(U),V,w,t}, & \text{if } p = \infty \end{cases} \tag{1.32}$$

and

$$|u|_{W_p^k(U),V,w,t} = \begin{cases} \Big(\displaystyle\sum_{|\alpha| = k} \|D^\alpha u\|_{L_p(U),V,w,t} \Big)^{1/p}, & \text{if } 1 \leq p < \infty \\ \displaystyle\max_{|\alpha| = k} \|D^\alpha u\|_{L_\infty(U),V,w,t}, & \text{if } p = \infty. \end{cases} \tag{1.33}$$

1.2 Elementary Inequalities

1.2.1 Scaling Inequalities

The following result concerns how the Sobolev seminorms of a function change under mappings which are nearly scalings. A straightforward proof is furnished by the chain rule and the change of variables formula.

Proposition 1.1. *Suppose that $1 \leq p \leq \infty$, $k \geq 0$ is an integer, U is an open subset of $\mathbb{R}^N$, $u \in W_p^k(U)$, and $c, d > 0$. Let $\Phi : U \to \mathbb{R}^N$ be invertible, $\hat{U} = \Phi(U)$, and $\hat{u} = u \circ \Phi^{-1}$.*

1. *If $|\Phi|_{(W_\infty^1(U))^N} \leq cd^{-1}$ and $|\Phi^{-1}|_{(W_\infty^i(\hat{U}))^N} \leq cd^i$ for all $i \in 1:k$ then*

$$|\hat{u}|_{W_p^k(\hat{U})} \leq Cd^{-N/p+k} \|u\|_{W_p^k(U)}, \tag{1.34}$$

   where C depends on N, k, and c.

2. *If $|\Phi|_{(W_\infty^1(U))^N} \leq cd^{-1}$, $|\Phi^{-1}|_{(W_\infty^1(\hat{U}))^N} \leq cd$, and $|\Phi^{-1}|_{(W_\infty^i(\hat{U}))^N} = 0$ for all $i \in 2:k$ then*

$$|\hat{u}|_{W_p^k(\hat{U})} \leq Cd^{-N/p+k} |u|_{W_p^k(U)}, \tag{1.35}$$

   where C depends on N, k, and c.

3. *If $D\Phi = dI$ then*

$$|\hat{u}|_{W_p^k(\hat{U})} = d^{-N/p+k} |u|_{W_p^k(U)}. \tag{1.36}$$

1.2.2 Negative Norm Inequalities

We state several properties of the general negative norm. These facts are trivial to verify.

Proposition 1.2. *Suppose that $1 \leq p \leq \infty$ and $k \geq 1$ is an integer.*

1. *If U, V are open subsets of $\mathbb{R}^N$ and $u \in L_p(U \cap V)$ then*

$$\|u\|_{W_p^{-k}(U\cap V)} \leq \|u\|_{W_p^{-k}(U,V)}. \tag{1.37}$$

2. *If U, V are open subsets of $\mathbb{R}^N$ with $U \subset V$ and $u \in L_p(U)$ then*

$$\|u\|_{W_p^{-k}(U,V)} = \|u\|_{W_p^{-k}(U)}. \tag{1.38}$$

3. *If U_1, U_2, V_1, V_2 are open subsets of $\mathbb{R}^N$, $U_1 \subset U_2$, $U_1 \cap V_1 = U_1 \cap V_2$, and $u \in L_p(U_2 \cap V_2)$ then*

$$\|u\|_{W_p^{-k}(U_1,V_1)} \leq \|u\|_{W_p^{-k}(U_2,V_2)}. \tag{1.39}$$

4. *If U_1, U_2 are open subsets of $\mathbb{R}^N$ with $U_1 \subset U_2$ and $u \in L_p(U_2)$ then*

$$\|u\|_{W_p^{-k}(U_1)} \leq \|u\|_{W_p^{-k}(U_2)}. \tag{1.40}$$

1.2.3 Sobolev's Inequalities

We single out two particular Sobolev inequalities.

Proposition 1.3. *Suppose that U is a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary.*

1. *If $u \in W_1^1(U)$ then*

$$\|u\|_{L_{\frac{N}{N-1}}(U)} \leq C\|u\|_{W_1^1(U)}, \tag{1.41}$$

where C depends on U and N.

2. *If $u \in W_{2N}^1(U)$ then*

$$\|u\|_{L_\infty(U)} \leq C\|u\|_{W_{2N}^1(U)}, \tag{1.42}$$

where C depends on U and N.

One unsatisfactory aspect of this is the untracked dependence on U. To remove this dependence, we map U, assumed to be of size roughly d, to a reference domain $\hat{U}$, of roughly unit size, and apply the Sobolev inequalities there. The result has an untracked dependence on the reference domain $\hat{U}$ and an explicit dependence on d. The scaling inequalities are used to translate the results obtained on the reference domain back to the original domain.

Corollary 1.4. *Suppose that U is a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary, $c > 0$, and let $d = \mathrm{diam}(U)$. Let $\Phi : U \to \mathbb{R}^N$ be invertible, $\hat{U} = \Phi(U)$, $|\Phi|_{(W^1_\infty(U))^N} \leq cd^{-1}$, and $|\Phi^{-1}|_{(W^1_\infty(\hat{U}))^N} \leq cd$.*

1. *If $u \in W^1_1(U)$ then*

$$\|u\|_{L_{\frac{N}{N-1}}(U)} \leq C\Big(d^{-1}\|u\|_{L_1(U)} + |u|_{W^1_1(U)}\Big), \tag{1.43}$$

where C depends on $\hat{U}$, N, and c.

2. *If $u \in W^1_{2N}(U)$ then*

$$\|u\|_{L_\infty(U)} \leq Cd^{1/2}\Big(d^{-1}\|u\|_{L_{2N}(U)} + |u|_{W^1_{2N}(U)}\Big), \tag{1.44}$$

where C depends on $\hat{U}$, N, and c.

1.2.4 Measure Inequality

The following is such a widely-used consequence of Hölder's inequality that it deserves to be singled out.

Proposition 1.5. *If U is a bounded open subset of $\mathbb{R}^N$, $1 \leq p \leq q \leq \infty$, and $u \in L_q(U)$ then*

$$\|u\|_{L_p(U)} \leq (\mathrm{meas}_N(U))^{1/p-1/q}\|u\|_{L_q(U)}. \tag{1.45}$$

1.2.5 Young's Inequality

The first result below is Young's inequality for convolution and the second is a consequence. The third is a generalisation, the proof of which can be modelled after that of [1, Theorem 2.24].

Proposition 1.6. *Suppose that* $1 \le p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ *and* V, W *are open subsets of* $\mathbb{R}^N$. *Let* $U = \{x - y : x \in W, y \in V\}$.

1. *If* $u \in L_p(U)$ *and* $v \in L_q(V)$ *then*
$$\|u * v\|_{L_r(W)} \le \|u\|_{L_p(U)} \|v\|_{L_q(V)}. \tag{1.46}$$

2. *If* $\bar{u} \in L_p(U)$, $v \in L_q(V)$, $u : W \times V \to \mathbb{R}$, $w(x) = \int_V u(x, y) v(y)\, dy$ *for* $x \in W$, *and* $|u(x, y)| \le \bar{u}(x - y)$ *for* $x \in W$ *and* $y \in V$ *then*
$$\|w\|_{L_r(W)} \le \|\bar{u}\|_{L_p(U)} \|v\|_{L_q(V)}. \tag{1.47}$$

3. *If* $u(x, \cdot) \in L_p(V)$ *for all* $x \in W$, $u(\cdot, y) \in L_p(W)$ *for all* $y \in V$, $v \in L_q(V)$, *and* $w(x) = \int_V u(x, y) v(y)\, dy$ *for* $x \in W$ *then*
$$\|w\|_{L_r(W)} \le \left(\sup_{x \in W} \|u(x, \cdot)\|_{L_p(V)}^{p/q'} \right) \left(\sup_{y \in V} \|u(\cdot, y)\|_{L_p(W)}^{p/r} \right) \|v\|_{L_q(V)}. \tag{1.48}$$

1.2.6 Weighted Seminorm Inequality

The following proposition provides an estimate for the weighted seminorms. It is a generalisation of an intermediate result in the proof of the asymptotic error expansion inequalities of [19, Theorem 4.1].

Proposition 1.7. *Suppose that* U *is an open subset of* $\mathbb{R}^N$, $V, W \subset \mathbb{R}^N$, $w, c > 0$, $k \ge 0$ *is an integer,* $t \ge 0$, *and* $u \in W_\infty^{k + \lceil t \rceil}(U)$. *Assume that* U *is star-shaped with respect to* W *and that, if* $x \in U$ *then*
$$\operatorname{dist}(x, W) \le c(w + \operatorname{dist}(x, V)). \tag{1.49}$$

Then

$$|u|_{W^k_\infty(U),V,w,t} \le C\Big(\sum_{i=0}^{\lceil t\rceil-1} w^i|u|_{W^{k+i}_\infty(W)} + w^t|u|_{W^{k+\lceil t\rceil}_\infty(U)}\Big), \tag{1.50}$$

where C depends on N, $\operatorname{diam}(U)$*, k, $\lceil t\rceil$, and c.*

Proof. In this proof, let C denote different positive constants that depend on N, $\operatorname{diam}(U)$, k, $\lceil t\rceil$, and c. Let $x \in U$ and $|\alpha| = k$. Choose $y \in W$ such that $|x-y| \le 2\operatorname{dist}(x,W)$. From the definition of the Taylor polynomial, it is clear that

$$|T_y^{\lceil t\rceil-1}D^\alpha u(x)| \le C\sum_{i=0}^{\lceil t\rceil-1}|D^\alpha u|_{W^i_\infty(W)}|x-y|^i. \tag{1.51}$$

By Taylor's theorem,

$$|(D^\alpha u - T_y^{\lceil t\rceil-1}D^\alpha u)(x)| \le C|x-y|^{\lceil t\rceil}|D^\alpha u|_{W^{\lceil t\rceil}_\infty(U)}. \tag{1.52}$$

Observe that, if $i \in 0:\lceil t\rceil$ then

$$\begin{aligned}\sigma^t_{V,w}(x)|x-y|^i &\le C\sigma^t_{V,w}(x)|x-y|^{\min\{i,t\}}\\ &= C\left(\frac{|x-y|}{w+\operatorname{dist}(x,V)}\right)^{\min\{i,t\}} w^{\min\{i,t\}}\sigma^{\max\{0,t-i\}}_{V,w}(x).\end{aligned} \tag{1.53}$$

Also,

$$|x-y| \le 2\operatorname{dist}(x,W) \le C(w+\operatorname{dist}(x,V)). \tag{1.54}$$

Using the fact that $0 \le \sigma_{V,w} \le 1$, along with Equations 1.53 and 1.54, we obtain that, for $i \in 0:\lceil t\rceil$,

$$\sigma^t_{V,w}(x)|x-y|^i \le Cw^{\min\{i,t\}}. \tag{1.55}$$

Putting together Equations 1.51, 1.52, and 1.55, we see that

$$\begin{aligned}\sigma^t_{V,w}(x)|D^\alpha u(x)| &\le \sigma^t_{V,w}(x)|(T_y^{\lceil t\rceil-1}D^\alpha u)(x)|\\ &\quad + \sigma^t_{V,w}(x)|(D^\alpha u - T_y^{\lceil t\rceil-1}D^\alpha u)(x)|\\ &\le C\Big(\sum_{i=0}^{\lceil t\rceil-1} w^i|D^\alpha u|_{W^i_\infty(W)} + w^t|D^\alpha u|_{W^{\lceil t\rceil}_\infty(U)}\Big).\end{aligned} \tag{1.56}$$

The proposition follows by summing over all $|\alpha| = k$. ∎

CHAPTER 2

EXPLICIT CONSTANTS IN L_P-BASED ESTIMATES FOR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS SATISFYING GENERAL BOUNDARY CONDITIONS

2.1 Introduction and Statement of Results

Let $N \geq 2$ be an integer and let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. Let $m \geq 1$ be an integer and, for $j \in 1 : m$, let $m_j \geq 0$ be an integer. Define $k_0 = \max_{j\in 1:m}\{2m, m_j + 1\}$ and let $k \geq 0$ be an integer. For α a multiindex of length N with $|\alpha| \leq 2m$, let $a_\alpha : \bar{\Omega} \to \mathbb{R}$ be sufficiently smooth. For $j \in 1 : m$ and β a multiindex of length N with $|\beta| \leq m_j$, let $b_{j,\beta} : \partial\Omega \to \mathbb{R}$ be sufficiently smooth.

Define the differential operator L on functions $u : \Omega \to \mathbb{R}$ by

$$Lu = \sum_{|\alpha|\leq 2m} a_\alpha D^\alpha u. \tag{2.1}$$

We assume that L is uniformly elliptic. That is, there exists a constant $C_{\text{ell}} > 0$ such that, if $x \in \Omega$ and $\xi \in \mathbb{R}^N$ then

$$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq C_{\text{ell}}|\xi|^{2m}. \tag{2.2}$$

We also assume that L satisfies a certain algebraic root condition, as described in [2, pp. 704, 663] and [11, p. 74]. If $x \in \bar{\Omega}$ and $\xi, \eta \in \mathbb{R}^N$ are linearly independent, define $P_{x,\xi,\eta} : \mathbb{R} \to \mathbb{R}$ by

$$P_{x,\xi,\eta}(t) = \sum_{|\alpha|=2m} a_\alpha(x)(\xi + t\eta)^\alpha. \tag{2.3}$$

It is assumed that $P_{x,\xi,\eta}$ has has exactly m roots with positive imaginary parts, which we will denote by $r^+_{x,\xi,\eta,i}$ for $i \in 1 : m$.

For $j \in 1 : m$, define the boundary differential operator B_j on functions $u : \bar{\Omega} \to \mathbb{R}$ by

$$B_j u = \sum_{|\beta| \le m_j} b_{j,\beta} D^\beta u. \tag{2.4}$$

We assume that L and the B_j satisfy a certain algebraic complementing condition, as described in [2, pp. 704, 663] and [11, p. 74]. If $x \in \partial\Omega$, $\xi \in \mathbb{R}^N$ is such that $\xi \neq 0$ but $\xi^\mathsf{T} \nu_\Omega(x) = 0$, and $j \in 1 : m$, define $P_{j,x,\xi} : \mathbb{R} \to \mathbb{R}$ by

$$P_{j,x,\xi}(t) = \sum_{|\beta| = m_j} b_{j,\beta}(x)(\xi + t\nu_\Omega(x))^\beta. \tag{2.5}$$

Also define $P^+_{x,\xi} : \mathbb{R} \to \mathbb{R}$ by

$$P^+_{x,\xi}(t) = \prod_{i=1}^{m} (t - r^+_{x,\xi,\nu_\Omega(x),i}). \tag{2.6}$$

It is assumed that the $P_{j,x,\xi} \mod P^+_{x,\xi}$ for $j \in 1 : m$ are linearly independent.

We will let C denote different positive constants that depend on N, Ω, m, k, C_{ell}, various norms of the coefficients of the differential operators, and various quantities arising from the algebraic conditions on the differential operators.

The following five theorems are our main results.

Theorem 2.1. *If $1 < p < \infty$, $k \ge k_0$, $u \in W^k_p(\Omega)$, $Lu \in W^{k-2m}_p(\Omega)$, and, for each $j \in 1 : m$, $v_j \in W^{k-m_j}_p(\Omega)$ is such that $B_j u = v_j$ on $\partial\Omega$, then*

$$|u|_{W^k_p(\Omega)} \le C C_p^2 \Big(|Lu|_{W^{k-2m}_p(\Omega)} + \sum_{j=1}^{m} \|v_j\|_{W^{k-m_j}_p(\Omega)} + C_p^3 \|u\|_{W^{k-1}_p(\Omega)} \Big). \tag{2.7}$$

Theorem 2.2. *Assume that $m = 1$ and $m_1 \in 0 : 1$. If $1 < p < \infty$, $k \ge 2$, $u \in W^k_p(\Omega)$, $Lu \in W^{k-2}_p(\Omega)$, and $v_1 \in W^{k-m_1}_p(\Omega)$ is such that $B_1 u = v_1$ on $\partial\Omega$ then*

$$|u|_{W^k_p(\Omega)} \le C C_p \Big(|Lu|_{W^{k-2}_p(\Omega)} + \|v_1\|_{W^{k-m_1}_p(\Omega)} + \|u\|_{W^{k-1}_p(\Omega)} \Big). \tag{2.8}$$

Theorem 2.3. *Assume that $m = 1$ and $m_1 \in 0 : 1$. Furthermore, assume that, if $f \in C^0(\Omega)$ and $g_1 \in C^{m_1}(\partial\Omega)$, then the boundary-value problem*

$$\begin{aligned} Lu &= f \quad \text{on } \Omega \\ B_1 u &= g_1 \quad \text{on } \partial\Omega \end{aligned} \tag{2.9}$$

has a unique solution $u \in C^2(\Omega) \cap C^{m_1}(\bar{\Omega})$. If $1 < p < \infty$, $k \geq 2$, $u \in W_p^k(\Omega)$, and $B_1 u = 0$ on $\partial\Omega$ then

$$\|u\|_{W_p^k(\Omega)} \leq CC_p \|Lu\|_{W_p^{k-2}(\Omega)}. \tag{2.10}$$

Theorem 2.4. *Assume that $m = 1$, $m_1 \in 0 : 1$, and $k \geq 2$. Let $d > 0$ and let U, V be open subsets of $\mathbb{R}^N$ with $U \subset V$ and $\operatorname{dist}(U, \partial V) \geq d$.*

1. *If $\frac{N}{N-1} \leq p \leq 2N$, $u \in W_p^k(V \cap \Omega)$, $Lu = 0$ on $V \cap \Omega$, $B_1 u = 0$ on $V \cap \partial\Omega$, and $\ell \in 1 : k$ then*

$$\|u\|_{W_p^k(U\cap\Omega)} \leq Cd^{-(k-\ell)} \|u\|_{W_p^\ell(V\cap\Omega)}. \tag{2.11}$$

2. *If $1 \leq p \leq \frac{N}{N-1}$, $u \in W_p^k(V \cap \Omega)$, $Lu = 0$ on $V \cap \Omega$, $B_1 u = 0$ on $V \cap \partial\Omega$, and $\ell \in 2 : k$ then*

$$\|u\|_{W_p^k(U\cap\Omega)} \leq Cd^{-(k-\ell)} \|u\|_{W_p^\ell(V\cap\Omega)}. \tag{2.12}$$

3. *If $2N \leq p \leq \infty$, $u \in W_{2N}^k(V \cap \Omega)$, $Lu = 0$ on $V \cap \Omega$, $B_1 u = 0$ on $V \cap \partial\Omega$, and $\ell \in 2 : k$ then*

$$\|u\|_{W_p^{k-1}(U\cap\Omega)} \leq Cd^{-(k-\ell)} \|u\|_{W_p^{\ell-1}(V\cap\Omega)}. \tag{2.13}$$

Theorem 2.5. *Assume that $m_j + 1 \leq m$ for all $j \in 1 : m$. Furthermore, assume that, if $f \in C^0(\Omega)$ and $g_j \in C^{m_j}(\partial\Omega)$ for all $j \in 1 : m$, then the boundary-value problem*

$$\begin{aligned} Lu &= f \quad \text{on } \Omega \\ B_j u &= g_j \quad \text{on } \partial\Omega \qquad \text{for all } j \in 1 : m \end{aligned} \tag{2.14}$$

has a unique solution $u \in C^{2m}(\Omega) \cap C^{\max_{j\in 1:m} m_j}(\bar{\Omega})$. *Let* U, V *be open subsets of* Ω *such that, if* $x \in U$ *and* $y \in V$ *then* $d_1 \le |x-y| \le d_2$. *Suppose that* $1 \le p \le \infty$, $u \in W_p^k(\Omega)$, $Lu = 0$ *outside of* V, *and* $B_j u = 0$ *on* $\partial\Omega$ *for all* $j \in 1:m$.

1. *If* $2m - k = N$ *then*

$$|u|_{W_p^k(U)} \le C d_2^N \Big(1 + \log|\frac{1}{d_2}|\Big) \|Lu\|_{L_p(V)}. \tag{2.15}$$

2. *If* $2m - k > 0$ *and* $2m - k \ne N$ *then*

$$|u|_{W_p^k(U)} \le C d_2^{2m-k} \|Lu\|_{L_p(V)}. \tag{2.16}$$

3. *If* $2m - k = 0$ *and* $d_1 > 0$ *then*

$$|u|_{W_p^k(U)} \le C \log\frac{d_2}{d_1} \|Lu\|_{L_p(V)}. \tag{2.17}$$

4. *If* $2m - k < 0$ *and* $d_1 > 0$ *then*

$$|u|_{W_p^k(U)} \le C d_1^{-(k-2m)} \|Lu\|_{L_p(V)}. \tag{2.18}$$

2.2 Relationship to Prior Work

The classic L_p-based estimates for solutions of elliptic partial differential equations satisfying general boundary conditions are given in [2, Theorem 15.2]. In these estimates, the dependence on p is not made explicit.

Theorems 2.1 and 2.2 improve upon [2, Theorem 15.2] by making the dependence on p explicit. If we blindly trace the dependence on p through the proof of [2, Theorem 15.2], we obtain far poorer estimates than those of the present work. Theorem 2.1 is just as general as [2, Theorem 15.2]. Theorem 2.2 pertains only to second-order equations and boundary conditions of order at most one, but the result is sharper than that of Theorem 2.1.

Theorem 2.3 improves upon the estimate of [2, p. 706] by making the dependence on p explicit for problems with unique solutions. However, Theorem 2.3 pertains only to second-order equations and homogeneous boundary conditions of order at most one.

In [12, Theorem 9.13], W_p^2 estimates for solutions of second-order equations satisfying homogeneous Dirichlet boundary conditions are given. Again, the dependence on p is not made explicit. When the dependence on p is traced through the proof of [12, Theorem 9.13], a somewhat poorer estimate is obtained than that of Theorem 2.2.

In [29, Equation 2], W_p^2 estimates for unique solutions of second-order equations satisfying homogeneous Dirichlet boundary conditions are given for $p \geq 2$, and the dependence on p is made explicit. This is done by freezing the coefficients and applying a linear transformation so that the principal part of the differential operator is the Laplacian. Estimates near the boundary are obtained by locally flattening the boundary and odd reflection. The precise dependence on p here is obtained from the estimates for the Newtonian potential in [12, Theorems 9.8 and 9.9]. Some of the techniques in this proof are used in the present work and are crucial for obtaining sharper results than those found by simply tracing the dependence on p through the proofs of the estimates in [2, Theorem 15.2] and [12, Theorem 9.13].

In [9, Remark 5.3], it is stated that no explicit dependence on p is known for W_p^2 and W_p^3 estimates for unique solutions of second-order equations satisfying general first-order homogeneous boundary conditions.

Theorem 2.3 is stated without substantial proof or reference in several papers. We list some of these here.

The large p case is claimed in [24, p. 3] for homogeneous Dirichlet boundary conditions.

The $k = 2$, $1 < p \leq 2$ case is claimed in [25, Lemma 2.2] for co-normal derivative boundary conditions. However, as can be seen from [25, Equations 3.58 and 1.3], the result appears to be mistakenly applied to a problem with more general first-order boundary conditions. The case of general first-order boundary conditions seems substantially more difficult to handle than the case of co-normal deriviative boundary conditions.

The $k \geq 2$, $2 \leq p < \infty$ case is claimed in [19, Equation 1.7] and [20, Lemma 2.2] for two problems. One has co-normal derivative boundary conditions and the other has more general first-order boundary conditions.

Theorem 2.4 gives local estimates for solutions of second-order homogeneous equations satisfying homogeneous boundary conditions of order at most one.

Theorem 2.5 gives local estimates for unique solutions of homogeneous equations satisfying homogeneous boundary conditions. This is simply a convenient repackaging of the Green's function estimates of [17, p. 965].

2.3 Differential Operator Properties

In this section, we single out several facts about the differential operators.

Although the coefficients of the boundary differential operators need only be defined on the boundary of the domain, they may easily be extended into the interior of the domain. Local extensions are naturally obtained by locally flattening the boundary. Global extensions are obtained from the local extensions with a partition of unity. Thus we may think of $b_{j,\beta} : \bar{\Omega} \to \mathbb{R}$.

Freezing the coefficients of the differential operators at a point, we obtain constant coefficient operators, which are more amenable to analysis. For $x \in \Omega$, define the constant coefficient operator L_x on functions $u : \Omega \to \mathbb{R}$ by

$$L_x u = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha u. \tag{2.19}$$

For $x \in \partial\Omega$ and $j \in 1 : m$, define the constant coefficient operator $B_{j,x}$ on functions $u : \bar{\Omega} \to \mathbb{R}$ by

$$B_{j,x} u = \sum_{|\beta|=m_j} b_{j,\beta}(x) D^\beta u. \tag{2.20}$$

We now single out two propositions which we will use several times in proving our results.

Proposition 2.6. *If U is an open subset of Ω, $1 \leq p \leq \infty$, $|\gamma| = k$, and $u \in W_p^{2m+k}(U)$ then*

$$\|LD^\gamma u\|_{L_p(U)} \leq C\Big(|Lu|_{W_p^k(U)} + \|u\|_{W_p^{2m-1+k}(U)}\Big). \tag{2.21}$$

Proof. By the general Leibniz rule,

$$D^\gamma Lu - LD^\gamma u = \sum_{|\alpha|<2m+k} c_\alpha D^\alpha u, \tag{2.22}$$

where the $c_\alpha : \bar{\Omega} \to \mathbb{R}$ can be expressed in terms of γ and the coefficients of L. The proposition is immediate from this. □

Proposition 2.7. *If U is an open subset of Ω, $1 \leq p \leq \infty$, $k \geq 2m$, and $u \in W_p^k(U)$ then*

$$|u|_{W_p^k(U)} \leq C\Big(|Lu|_{W_p^{k-2m}(U)} + \max_{\substack{|\zeta|=k-2m \\ \zeta_N=0}} |D^\zeta u|_{W_p^{2m}(U)} + \|u\|_{W_p^{k-1}(U)}\Big). \tag{2.23}$$

Proof. Notice that $\|D^\eta u\|_{L_p(U)}$ is bounded by the right side of Equation 2.23, for all $|\eta| = k$ with $\eta_N \in 0 : 2m$. In this case, there exist $|\gamma| = 2m$ and $|\zeta| = k - 2m$ such that $\eta = \gamma + \zeta$, $\gamma_N = \eta_N$, and $\zeta_N = 0$.

We proceed by induction, following the proof of [10, Theorem 6.3.5]. Assume that $\|D^\eta u\|_{L_p(U)}$ is bounded by the right side of Equation 2.23 for all $|\eta| = k$ with

$\eta_N \in 0 : i$, for some $i \in 2m : k-1$. Then let $|\eta| = k$ and $\eta_N = i+1$. Since $\eta_N \geq 2m$, we can write $\eta = 2me_N + \zeta$, where $|\zeta| = k - 2m$ and $\zeta_N = i + 1 - 2m$. Obseve that

$$LD^\zeta u = \sum_{\substack{|\alpha| \leq 2m \\ \alpha_N < 2m}} a_\alpha D^{\alpha+\zeta} u + a_{2me_N} D^{2me_N+\zeta} u. \tag{2.24}$$

If $x \in U$ then, by uniform ellipticity,

$$\begin{aligned} a_{2me_N}(x) &= \sum_{|\alpha|=2m} a_\alpha(x) e_N^\alpha \\ &\geq C|e_N|^2 \\ &= C. \end{aligned} \tag{2.25}$$

Therefore we can divide both sides of Equation 2.24 by a_{2me_N}, yielding

$$D^\eta u = \frac{1}{a_{2me_N}} \Big(LD^\zeta u - \sum_{\substack{|\alpha| \leq 2m \\ \alpha_N < 2m}} a_\alpha D^{\alpha+\zeta} u \Big). \tag{2.26}$$

If $|\alpha| = 2m$ and $\alpha_N < 2m$ then $|\alpha + \zeta| = k$ and $(\alpha+\zeta)_N \leq i$. Therefore, by the induction hypothesis, $\|D^{\alpha+\zeta}u\|_{L_p(U)}$ is bounded by the right side of Equation 2.23. By Proposition 2.6, $\|LD^\zeta u\|_{L_p(U)}$ is bounded by the right side of Equation 2.23. These facts, together with Equations 2.26 and 2.25, show that $D^\eta u$ is bounded by the right side of Equation 2.23. $\square$

2.4 Operator Transformations

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be invertible and sufficiently smooth and have sufficiently smooth inverse, and let $\hat{\Omega} = \Phi(\Omega)$. In this section, we investigate how the differential operators transform under Φ and show that all our assumptions are preserved.

Define the transformed operator $\hat{L}$ on functions $\hat{u} : \hat{\Omega} \to \mathbb{R}$ by

$$\hat{L}\hat{u} = \sum_{|\alpha| \leq 2m} \hat{a}_\alpha D^\alpha \hat{u}, \tag{2.27}$$

where the coefficients $\hat{a}_\alpha : \bar{\hat{\Omega}} \to \mathbb{R}$ are such that

$$\hat{L}\hat{u} = L(\hat{u} \circ \Phi) \circ \Phi^{-1}. \tag{2.28}$$

For $j \in 1 : m$, define the transformed operator $\hat{B}_j$ on functions $\hat{u} : \bar{\hat{\Omega}} \to \mathbb{R}$ by

$$\hat{B}_j \hat{u} = \sum_{|\beta| \le m_j} \hat{b}_{j,\beta} D^\beta \hat{u}, \tag{2.29}$$

where the coefficients $\hat{b}_{j,\beta} : \bar{\hat{\Omega}} \to \mathbb{R}$ are such that

$$\hat{B}_j \hat{u} = \hat{B}_j(\hat{u} \circ \Phi) \circ \Phi^{-1}. \tag{2.30}$$

Fix $\hat{x} \in \hat{\Omega}$ and let $x = \Phi^{-1}(\hat{x})$. We know by [4, Section 3.9] that the normal vector transforms according to

$$\nu_\Omega(x) = (D\Phi(x))^\mathsf{T} \nu_{\hat{\Omega}}(\hat{x}). \tag{2.31}$$

Fix $\hat{\xi} \in \mathbb{R}^N$ and let $\xi = (D\Phi(x))^\mathsf{T} \hat{\xi}$. Define $\hat{u} : \hat{\Omega} \to \mathbb{R}$ by

$$\hat{u}(\hat{y}) = \sum_{|\alpha| = 2m} \frac{1}{\alpha!} \xi^\alpha (\hat{y} - \hat{x})^\alpha. \tag{2.32}$$

It is easily computed that

$$D^\alpha \hat{u}(\hat{x}) = \begin{cases} \hat{\xi}^\alpha, & \text{if } |\alpha| = 2m \\ 0, & \text{otherwise.} \end{cases} \tag{2.33}$$

Now let $u = \hat{u} \circ \Phi$. If $|\alpha| = 2m$ then, writing $\alpha = e_{i_1} + \cdots + e_{i_{2m}}$, we see that

$$\begin{aligned} D^\alpha u(x) &= \sum_{j_1=1}^N \cdots \sum_{j_{2m}=1}^N (D\Phi(x))_{j_1, i_i} \cdots (D\Phi(x))_{j_{2m}, i_{2m}} \hat{\xi}_{j_1} \cdots \hat{\xi}_{j_{2m}} \\ &= ((D\Phi(x))^\mathsf{T} \hat{\xi})_{i_1} \cdots ((D\Phi(x))^\mathsf{T} \hat{\xi})_{i_{2m}} \\ &= ((D\Phi(x))^\mathsf{T} \hat{\xi})^\alpha \\ &= \xi^\alpha. \end{aligned} \tag{2.34}$$

If $|\alpha| < 2m$ then $D^\alpha u(x) = 0$. By Equations 2.33, 2.27, 2.28, and 2.34,

$$\begin{aligned}\sum_{|\alpha|=2m} \hat{a}_\alpha(\hat{x})\hat{\xi}^\alpha &= \hat{L}\hat{u}(\hat{x}) \\ &= Lu(x) \\ &= \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha.\end{aligned} \tag{2.35}$$

A similar argument establishes that, for $j \in 1 : m$,

$$\sum_{|\beta|=m_j} \hat{b}_{j,\beta}(\hat{x})\hat{\xi}^\beta = \sum_{|\beta|=m_j} b_{j,\beta}(x)\xi^\beta. \tag{2.36}$$

Using Equation 2.35, it easily verified that uniform ellipticity and the root condition are preserved under transformation. Using Equations 2.36 and 2.31, it is easily verified that the complementing condition is preserved under transformation.

2.5 Estimates in the Interior

Lemma 2.8. *Suppose that $x_0 \in \Omega$ and $d > 0$. Let $U = B_d(x_0)$ and $V = B_{3d}(x_0)$ and assume that $V \subset \Omega$. If $1 < p < \infty$, $k \geq k_0$, and $u \in W_p^k(V)$ then*

$$\begin{aligned}|u|_{W_p^k(U)} \leq C\Big(&C_p|Lu|_{W_p^{k-2m}(V)} \\ &+ C_p d|u|_{W_p^k(V)} \\ &+ (C_p + d^{-1})\|u\|_{W_p^{k-1}(V)}\Big).\end{aligned} \tag{2.37}$$

Proof. For $i \in 1 : 3$, let $U_i = B_{id}(x_0)$.

By the Bramble-Hilbert lemma, there exists some $\chi \in \Pi^{k-2}(U_3)$ such that, if $i \in 0 : k-1$ then

$$|u - \chi|_{W_p^i(U_3)} \leq Cd^{k-1-i}|u|_{W_p^{k-1}(U_3)}. \tag{2.38}$$

Let $\omega \in C_0^\infty(U_3)$ be such that $\omega = 1$ on U_2 and, for $i \in 0 : k$,

$$|\omega|_{W_\infty^i(U_3)} \leq Cd^{-i}. \tag{2.39}$$

Let $|\gamma| = k$. Then there exist $|\zeta| = 2m$ and $|\eta| = k - 2m$ such that $\gamma = \zeta + \eta$. By the general Leibniz rule,

$$D^\eta L_{x_0}(\omega(u - \chi)) = g + h, \tag{2.40}$$

where

$$g = \omega D^\eta\Big(Lu - \sum_{|\alpha|=2m}(a_\alpha - a_\alpha(x_0))D^\alpha u - \sum_{|\alpha|<2m} a_\alpha D^\alpha u\Big) \tag{2.41}$$

and

$$h = \sum_{\substack{|\alpha|+|\beta|=k \\ |\alpha|>0}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta(u - \chi), \tag{2.42}$$

and the $c_{1,\alpha,\beta}$ are constants that depend on η and the coefficients of L_{x_0}.

Let $\Gamma : \mathbb{R}^N \smallsetminus \{0\} \to \mathbb{R}$ denote the fundamental solution corresponding to the constant coefficient operator L_{x_0}, as given in [16, pp. 69–70] and described in [3, p. 213] and [2, p. 652]. By [3, Chapter 5, Equation 5],

$$\omega(u - \chi) = \Gamma * L_{x_0}(\omega(u - \chi)). \tag{2.43}$$

By [3, Chapter 5, Equation 26], along with Equations 2.43 and 2.40,

$$D^\gamma(\omega(u - \chi)) = D^\zeta\Gamma \,\hat{*}\, (g + h) + c_2(g + h), \tag{2.44}$$

where c_2 is a constant that depends on ζ and the coefficients of L_{x_0}.

By [2, Equation 4.2], we see that the $(2m - 1)$st-order derivatives of Γ are homogeneous of degree $-(N - 1)$. Notice that $h = 0$ outside of $U_3 \smallsetminus U_2$ and that, if $x \in U_1$ and $y \in U_3 \smallsetminus U_2$ then $d \le |x - y| \le 4d$. Therefore, by Corollary 2.25, Part 1,

$$\|D^\zeta\Gamma \,\hat{*}\, (g + h)\|_{L_p(U_1)} \le C\Big(C_p\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.45}$$

By Equations 2.44 and 2.45,

$$\|D^\gamma(\omega(u - \chi))\|_{L_p(U_1)} \le C\Big(C_p\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.46}$$

It remains to estimate $\|g\|_{L_p(U_3)}$ and $\|h\|_{L_p(U_3)}$.

By Equation 2.41,

$$\|g\|_{L_p(U_3)} \leq C\Big(|Lu|_{W_p^{k-2m}(U_3)} + d|u|_{W_p^k(U_3)} + \|u\|_{W_p^{k-1}(U_3)}\Big). \tag{2.47}$$

If $|\alpha| + |\beta| = k$ and $|\beta| < k$ then, by Equations 2.38 and 2.39,

$$\begin{aligned} \|D^\alpha \omega D^\beta (u-\chi)\|_{L_p(U_3)} &\leq |\omega|_{W_\infty^{|\alpha|}(U_3)} |u-\chi|_{W_p^{|\beta|}(U_3)} \\ &\leq C d^{-|\alpha|} d^{(k-1)-|\beta|} |u|_{W_p^{k-1}(U_3)} \\ &= C d^{-1} |u|_{W_p^{k-1}(U_3)}. \end{aligned} \tag{2.48}$$

By Equations 2.42 and 2.48,

$$\|h\|_{L_p(U_3)} \leq C d^{-1} |u|_{W_p^{k-1}(U_3)}. \tag{2.49}$$

Putting together Equations 2.46, 2.47, and 2.49,

$$\begin{aligned} \|D^\gamma u\|_{L_p(U_1)} &= \|D^\gamma(\omega(u-\chi))\|_{L_p(U_1)} \\ &\leq C\Big(C_p |Lu|_{W_p^{k-2m}(U_3)} \\ &\quad + C_p d |u|_{W_p^k(U_3)} \\ &\quad + (C_p + d^{-1}) \|u\|_{W_p^{k-1}(U_3)}\Big). \end{aligned} \tag{2.50}$$

The lemma follows by summing this inequality over all $|\gamma| = k$. □

2.6 Estimates at the Boundary

2.6.1 The General Case

First we handle the case of a flat boundary portion.

Lemma 2.9. *Suppose that $x_0 \in \partial\Omega$ and $d > 0$. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{3d}(x_0) \cap \Omega$, and $T = B_{3d}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$. If*

$1 < p < \infty$, $k \geq k_0$, $u \in W_p^k(V^+)$ and, for each $j \in 1 : m$, $v_j \in W_p^{k-m_j}(V^+)$ is such that $B_j u = v_j$ on T, then

$$\begin{aligned}|u|_{W_p^k(U^+)} \leq C\Big(& C_p^2 |Lu|_{W_p^{k-2m}(V^+)} \\ & + (C_p + d^{-1}) \sum_{j=1}^{m} \|v_j\|_{W_p^{k-m_j}(V^+)} \\ & + C_p^2 d |u|_{W_p^k(V^+)} \\ & + C_p d^{-1}(C_p + d^{-1}) \|u\|_{W_p^{k-1}(V^+)}\Big).\end{aligned} \tag{2.51}$$

Proof. For $i \in 1 : 3$, let $U_i = B_{id}(x_0)$ and $U_i^+ = B_{id}(x_0) \cap \mathbb{R}_+^N$.

By the extension theorem, there exists an extension $\bar{u} : U_3 \to \mathbb{R}$ of u such that, for $i \in 0 : k$,

$$|\bar{u}|_{W_p^i(U_3)} \leq C |u|_{W_p^i(U_3^+)}. \tag{2.52}$$

Although it is not explicitly stated, it is easily seen in the proof of [1, Theorem 5.19] that this constant does not depend on p or d.

By the Bramble-Hilbert lemma, there exists some $\chi \in \Pi^{k-2}(U_3)$ such that, if $i \in 0 : k-1$ then

$$|\bar{u} - \chi|_{W_p^i(U_3)} \leq C d^{k-1-i} |\bar{u}|_{W_p^{k-1}(U_3)}. \tag{2.53}$$

Let $\omega \in C_0^\infty(U_3)$ be such that $\omega = 1$ on U_2 and, for $i \in 0 : k$,

$$|\omega|_{W_\infty^i(U_3)} \leq C d^{-i}. \tag{2.54}$$

Ideally, we would be able to proceed, as in Section 2.5, by analysing a representation of $\omega(\bar{u} - \chi)$ in terms of $L_{x_0}(\omega(\bar{u} - \chi))$. This will not work here, however, because it appears impossible to bound the extension of Lu in terms of Lu. A proposed method like this would also be suspect because it completely ignores the boundary conditions.

Here we start by finding a function $v : \bar{\mathbb{R}}_+^N \to \mathbb{R}$ such that $\omega(\bar{u} - \chi) - v$ is

L_{x_0}-harmonic on $\mathbb{R}^N_+$. By the general Leibniz rule,

$$\begin{aligned} L_{x_0}(\omega(u-\chi)) = \omega\Bigg(& L(u-\chi) \\ & - \sum_{|\alpha|=2m} (a_\alpha - a_\alpha(x_0)) D^\alpha (u-\chi) \\ & - \sum_{|\alpha|<2m} a_\alpha D^\alpha (u-\chi) \Bigg) \\ & + \sum_{\substack{|\alpha|+|\beta|=2m \\ |\alpha|>0}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (u-\chi), \end{aligned} \tag{2.55}$$

where the $c_{1,\alpha,\beta}$ are constants that depend on the coefficients of L_{x_0}. By the extension theorem, there exists an extension $\bar{f} : U_3 \to \mathbb{R}$ of $L(u-\chi)$ such that, for $i \in 0 : k-2m$,

$$|\bar{f}|_{W^i_p(U_3)} \leq C |L(u-\chi)|_{W^i_p(U_3^+)}. \tag{2.56}$$

Also by the extension theorem, there exist sufficiently smooth extensions $\bar{a}_\alpha : U_3 \to \mathbb{R}$ of a_α for $|\alpha| \leq 2m$. Now define $F : U_3 \to \mathbb{R}$ by

$$\begin{aligned} F = \omega\Bigg(& \bar{f} \\ & - \sum_{|\alpha|=2m} (\bar{a}_\alpha - \bar{a}_\alpha(x_0)) D^\alpha (\bar{u}-\chi) \\ & - \sum_{|\alpha|<2m} \bar{a}_\alpha D^\alpha (\bar{u}-\chi) \Bigg) \\ & + \sum_{\substack{|\alpha|+|\beta|=2m \\ |\alpha|>0}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (\bar{u}-\chi), \end{aligned} \tag{2.57}$$

and notice that, by Equation 2.55, on U_3^+,

$$F = L_{x_0}(\omega(u-\chi)). \tag{2.58}$$

Let $\Gamma : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ denote the fundamental solution corresponding to the constant coefficient operator L_{x_0} and define $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ by

$$v = \Gamma * F. \tag{2.59}$$

By Equation 2.58 and [3, Chapter 5, Equation 5], we see that, on U_3^+,

$$L_{x_0} v = L_{x_0}(\omega(u - \chi)). \tag{2.60}$$

Let $|\gamma| = k$. Then there exist $|\zeta| = 2m$ and $|\eta| = k - 2m$ such that $\gamma = \zeta + \eta$. By the general Leibniz rule,

$$D^\eta F = g + h, \tag{2.61}$$

where

$$g = \omega\Big(D^\eta f - \sum_{|\alpha|=2m} (\bar{a}_\alpha - \bar{a}_\alpha(x_0)) D^{\eta+\alpha} \bar{u} + \sum_{|\alpha|<k} c_{3,\alpha} D^\alpha (\bar{u} - \chi)\Big) \tag{2.62}$$

and

$$h = \sum_{\substack{|\alpha|+|\beta|=k-2m \\ |\alpha|>0}} c_{4,\alpha,\beta} D^\alpha \omega D^\beta \bar{f} + \sum_{\substack{|\alpha|+|\beta|\le k \\ |\alpha|>0}} c_{5,\alpha,\beta} D^\alpha \omega D^\beta (\bar{u} - \chi), \tag{2.63}$$

and the $c_{3,\alpha}$, $c_{4,\alpha,\beta}$, and $c_{5,\alpha,\beta}$ are constants that depend on η and the coefficients of L_{x_0}. By [3, Chapter 5, Equation 26], along with Equations 2.59 and 2.61,

$$D^\gamma v = D^\zeta \Gamma \,\hat{*}\, (g + h) + c_6 (g + h), \tag{2.64}$$

where c_6 is a constant that depends on ζ and the coefficients of L_{x_0}.

By [2, Equation 4.2], we see that the $(2m-1)$st-order derivatives of Γ are homogeneous of degree $-(N-1)$. Notice that $h = 0$ outside of $U_3 \setminus U_2$ and that, if $x \in U_1^+$ and $y \in U_3 \setminus U_2$ then $d \le |x - y| \le 4d$. Therefore, by Corollary 2.25, Part 1,

$$\|D^\zeta \Gamma \,\hat{*}\, (g + h)\|_{L_p(U_1^+)} \le C\Big(C_p \|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.65}$$

By Equations 2.64 and 2.65,

$$\|D^\gamma v\|_{L_p(U_1^+)} \le C\Big(C_p \|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.66}$$

We will need an estimate for $D^\gamma v$ on $\mathbb{R}^N_+$ in addition to this one on U_1^+. By Theorem 2.23,

$$\|D^\zeta \Gamma \hat{*} (g+h)\|_{L_p(\mathbb{R}^N_+)} \le CC_p\Big(\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.67}$$

By Equations 2.64 and 2.67,

$$\|D^\gamma v\|_{L_p(\mathbb{R}^N_+)} \le CC_p\Big(\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big). \tag{2.68}$$

It remains to estimate $\|g\|_{L_p(U_3)}$ and $\|h\|_{L_p(U_3)}$.

By Equation 2.62,

$$\|g\|_{L_p(U_3)} \le C\Big(|\bar{f}|_{W_p^{k-2m}(U_3)} + d|\bar{u}|_{W_p^k(U_3)} + \|\bar{u}\|_{W_p^{k-1}(U_3)}\Big). \tag{2.69}$$

By Equation 2.56,

$$\begin{aligned} |\bar{f}|_{W_p^{k-2m}(U_3)} &\le C|L(u-\chi)|_{W_p^{k-2m}(U_3^+)} \\ &\le C\Big(|Lu|_{W_p^{k-2m}(U_3^+)} + |L\chi|_{W_p^{k-2m}(U_3^+)}\Big). \end{aligned} \tag{2.70}$$

Using the fact that χ is a polynomial of degree at most $k-1$, along with Equations 2.53 and 2.52,

$$\begin{aligned} |L\chi|_{W_p^{k-2m}(U_3^+)} &\le C\|\chi\|_{W_p^{k-1}(U_3^+)} \\ &\le C\Big(\|u-\chi\|_{W_p^{k-1}(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)}\Big) \\ &\le C\|u\|_{W_p^{k-1}(U_3^+)}. \end{aligned} \tag{2.71}$$

Putting together Equations 2.69, 2.70, 2.71, and 2.52 yields

$$\|g\|_{L_p(U_3^+)} \le C\Big(|Lu|_{W_p^{k-2m}(U_3^+)} + d|u|_{W_p^k(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)}\Big). \tag{2.72}$$

Suppose first that $|\alpha| + |\beta| = k - 2m$ and $|\alpha| > 0$. By Equations 2.56, 2.53, and 2.52,

$$\begin{aligned} |\bar{f}|_{W_p^{|\beta|}(U_3)} &\le C|L(u-\chi)|_{W_p^{|\beta|}(U_3^+)} \\ &\le C\|u-\chi\|_{W_p^{|\beta|+2m}(U_3^+)} \\ &\le Cd^{(k-1)-(|\beta|+2m)}|u|_{W_p^{k-1}(U_3^+)}, \end{aligned} \tag{2.73}$$

so, by Equation 2.54,

$$\begin{aligned}\|D^\alpha\omega D^\beta\bar{f}\|_{L_p(U_3)} &\le |\omega|_{W_\infty^{|\alpha|}(U_3)}|\bar{f}|_{W_p^{|\beta|}(U_3)}\\ &\le Cd^{-|\alpha|}d^{(k-1)-(|\beta|+2m)}|u|_{W_p^{k-1}(U_3^+)}\\ &= Cd^{-1}|u|_{W_p^{k-1}(U_3^+)}.\end{aligned}\tag{2.74}$$

Next, if $|\alpha|+|\beta|\le k$ and $|\alpha|>0$ then, by Equations 2.54, 2.53, and 2.52,

$$\begin{aligned}\|D^\alpha\omega D^\beta(\bar{u}-\chi)\|_{L_p(U_3)} &\le |\omega|_{W_\infty^{|\alpha|}(U_3)}\|\bar{u}-\chi\|_{W_p^{|\beta|}(U_3)}\\ &\le Cd^{-|\alpha|}d^{(k-1)-|\beta|}\|u\|_{W_p^{k-1}(U_3^+)}\\ &= Cd^{-1}|u|_{W_p^{k-1}(U_3^+)}.\end{aligned}\tag{2.75}$$

Putting together Equations 2.63, 2.74, and 2.75,

$$\|h\|_{L_p(U_3)} \le Cd^{-1}\|u\|_{W_p^{k-1}(U_3^+)}.\tag{2.76}$$

Equations 2.66, 2.72 and 2.76 show that

$$\begin{aligned}\|D^\gamma v\|_{L_p(U_1^+)} \le C\Big(&C_p|Lu|_{W_p^{k-2m}(U_3^+)}\\ &+ C_pd|u|_{W_p^k(U_3^+)}\\ &+ (C_p+d^{-1})\|u\|_{W_p^{k-1}(U_3^+)}\Big),\end{aligned}\tag{2.77}$$

and Equations 2.68, 2.72 and 2.76 show that

$$\|D^\gamma v\|_{L_p(\mathbb{R}_+^N)} \le CC_p\Big(|Lu|_{W_p^{k-2m}(U_3^+)} + d|u|_{W_p^k(U_3^+)} + d^{-1}\|u\|_{W_p^{k-1}(U_3^+)}\Big).\tag{2.78}$$

Summing Equations 2.77 and 2.78 over all $|\gamma|=k$, we find that

$$\begin{aligned}|v|_{W_p^k(U_1^+)} \le C\Big(&C_p|Lu|_{W_p^{k-2m}(U_3^+)}\\ &+ C_pd|u|_{W_p^k(U_3^+)}\\ &+ (C_p+d^{-1})\|u\|_{W_p^{k-1}(U_3^+)}\Big)\end{aligned}\tag{2.79}$$

and

$$|v|_{W_p^k(\mathbb{R}_+^N)} \le CC_p\Big(|Lu|_{W_p^{k-2m}(U_3^+)} + d|u|_{W_p^k(U_3^+)} + d^{-1}\|u\|_{W_p^{k-1}(U_3^+)}\Big).\tag{2.80}$$

So far, we have constructed a function $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ such that $\omega(u - \chi) - v$ is L_{x_0}-harmonic on $\mathbb{R}^N_+$, and we have obtained estimates for it. As we will see, we will be able to obtain sharper estimates if, in addition, $B_{j,x_0} v = 0$ on $\partial\mathbb{R}^N_+$ for all $j \in 1 : m$. Such a construction seems most plausible for $m = 1$. This possibility is investigated in Section 2.6.2.

A representation of L_{x_0}-harmonic functions which satisfy general boundary conditions is given in [2, Section 2]. We use this to obtain estimates for $\omega(u-\chi)-v$, which, combined with the estimates for v that we have already demonstrated, yield estimates for $\omega(u - \chi)$.

Once again let $|\gamma| = k$. By [2, Theorem 4.1 and Corollary to Theorem 14.1], on $\mathbb{R}^N_+$,

$$D^\gamma(\omega(u - \chi)) = D^\gamma v + \sum_{j=1}^{m} \sum_{i=1}^{N} \sum_{\substack{|\eta|=k-m_j-1 \\ \eta_N=0}} D_i I_{i,j,\eta}, \tag{2.81}$$

where, by [2, Equations 4.13 and 4.13′], for $x \in \mathbb{R}^{N-1}$ and $t > 0$,

$$I_{i,j,\eta}(x, t) = \int_{\mathbb{R}^{N-1}} K_{i,j}(x - y, t) D^\eta B_{j,x_0}(\omega(u - \chi))(y, 0)\, dy. \tag{2.82}$$

Here, by [2, Lemma 2.1], the $K_{i,j}$ are sums of terms $K \in C^\infty(\mathbb{R}^N_+)$ which are homogeneous of degree $-(N-1)$ and satisfy $\|K\|_{W^2_\infty(\Sigma^{N-1}_+)} \leq C$, and, by [2, Equation 3.15], have the property that

$$\int_{\Sigma^{N-2}} K(x, 0)\, dS(x) = 0. \tag{2.83}$$

Let $j \in 1 : m$ and let $|\eta| = k - m_j - 1$ have $\eta_N = 0$. By the general Leibniz rule, on $\partial\mathbb{R}^N_+$,

$$
\begin{aligned}
D^\eta B_{j,x_0}(\omega(u-\chi)-v) = \omega D^\eta \Bigg(& B_j u \\
& - \sum_{|\beta|=m_j} (b_{j,\beta}-b_{j,\beta}(x_0))D^\beta u - \sum_{|\beta|<m_j} b_{j,\beta} D^\beta u \Bigg) \\
& + \sum_{\substack{|\alpha|+|\beta|=k-1 \\ |\alpha|>0}} c_{7,\alpha,\beta} D^\alpha \omega D^\beta (u-\chi) \\
& - D^\eta B_{j,x_0} v,
\end{aligned}
\tag{2.84}
$$

where the $c_{7,\alpha,\beta}$ are constants that depend on η and the coefficients of B_{j,x_0}. Now define $G : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ by

$$
\begin{aligned}
G = \omega D^\eta \Bigg(& v_j \\
& - \sum_{|\beta|=m_j} (b_{j,\beta}-b_{j,\beta}(x_0))D^\beta u - \sum_{|\beta|<m_j} b_{j,\beta} D^\beta u \Bigg) \\
& + \sum_{\substack{|\alpha|+|\beta|=k-1 \\ |\alpha|>0}} c_{7,\alpha,\beta} D^\alpha \omega D^\beta (u-\chi) \\
& - D^\eta B_{j,x_0} v,
\end{aligned}
\tag{2.85}
$$

and notice that, by Equation 2.84, on $\partial\mathbb{R}^N_+$,

$$
G = D^\eta B_{j,x_0}(\omega(u-\chi)-v). \tag{2.86}
$$

By the general Leibniz rule, if $\ell \in 1 : N$ then

$$
D_\ell G = g_\ell + h_\ell, \tag{2.87}
$$

where

$$
\begin{aligned}
g_\ell = \omega D^{\eta+e_\ell} \Bigg(& v_j - \sum_{|\beta|=m_j} (b_{j,\beta}-b_{j,\beta}(x_0))D^\beta u - \sum_{|\beta|<m_j} b_{j,\beta} D^\beta u \Bigg) \\
& - D^{\eta+e_\ell} B_{j,x_0} v
\end{aligned}
\tag{2.88}
$$

and

$$h_\ell = D_\ell \omega \Bigg(D^\eta v_j + \sum_{|\beta|<k} c_{8,\beta} D^\beta u \Bigg) + \sum_{\substack{|\alpha|+|\beta|=k \\ |\alpha|>0}} c_{9,\alpha,\beta} D^\alpha \omega D^\beta (u - \chi), \tag{2.89}$$

and the $c_{8,\beta}$ and $c_{9,\alpha,\beta}$ are constants that depend on η, ℓ, and the coefficients of B_{j,x_0}. The last term on the right side of Equation 2.88 is the most damaging in the estimates that follow.

Notice that $h_\ell = 0$ outside of $U_3^+ \setminus U_2^+$ and that, if $x \in U_1^+$ and $y \in U_3^+ \setminus U_2^+$ then $d \le |x - y^*| \le 4d$. Therefore, by Equations 2.82, 2.86, and 2.87, along with Theorem 2.29,

$$|I_{i,j,\eta}|_{W_p^1(U_1^+)} \le C \sum_{\ell=1}^{N} \Big(C_p \|g_\ell\|_{L_p(\mathbb{R}_+^N)} + \|h_\ell\|_{L_p(U_3^+)} \Big). \tag{2.90}$$

It remains to estimate $\|g_\ell\|_{L_p(\mathbb{R}_+^N)}$ and $\|h_\ell\|_{L_p(U_3^+)}$.

By Equation 2.88,

$$\begin{aligned} \|g_\ell\|_{L_p(\mathbb{R}_+^N)} \le C \Big(& |v_j|_{W_p^{k-m_j}(U_3^+)} + d|u|_{W_p^k(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)} \\ & + |v|_{W_p^k(\mathbb{R}_+^N)} \Big). \end{aligned} \tag{2.91}$$

If $|\alpha| + |\beta| = k$ and $|\alpha| > 0$ then, by Equation 2.75,

$$\begin{aligned} \|D^\alpha \omega D^\beta (u - \chi)\|_{L_p(U_3^+)} &\le \|D^\alpha \omega D^\beta (\bar{u} - \chi)\|_{L_p(U_3)} \\ &\le C d^{-1} |u|_{W_p^{k-1}(U_3^+)}. \end{aligned} \tag{2.92}$$

By Equations 2.89 and 2.92,

$$\|h_\ell\|_{L_p(U_3^+)} \le d^{-1} \Big(|v_j|_{W_p^{k-m_j-1}(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)} \Big). \tag{2.93}$$

Equations 2.90, 2.91, and 2.93 show that

$$\begin{aligned} |I_{i,j,\eta}|_{W_p^1(U_1^+)} \le C \Big(& (C_p + d^{-1}) \|v_j\|_{W_p^{k-m_j}(U_3^+)} \\ & + C_p d |u|_{W_p^k(U_3^+)} \\ & + (C_p + d^{-1}) \|u\|_{W_p^{k-1}(U_3^+)} \\ & + C_p |v|_{W_p^k(\mathbb{R}_+^N)} \Big). \end{aligned} \tag{2.94}$$

The triple sum on the right side of Equation 2.81 is estimated by summing Equation 2.94 first over all $|\eta| = k - m_j - 1$ with $\eta_N = 0$ and then over all $j \in 1 : m$. The result is that

$$\begin{aligned}\|D^\gamma(\omega(u-\chi))\|_{L_p(U_1^+)} \le C\Big(&|v|_{W_p^k(U_1^+)} \\ &+ (C_p + d^{-1})\sum_{j=1}^{m}\|v_j\|_{W_p^{k-m_j}(U_3^+)} \\ &+ C_p d|u|_{W_p^k(U_3^+)} \\ &+ (C_p + d^{-1})\|u\|_{W_p^{k-1}(U_3^+)} \\ &+ C_p|v|_{W_p^k(\mathbb{R}_+^N)}\Big).\end{aligned} \tag{2.95}$$

Using Equation 2.79 to estimate the first term and Equation 2.80 to estimate the last term, we obtain

$$\begin{aligned}\|D^\gamma u\|_{L_p(U_1^+)} &= \|D^\gamma(\omega(u-\chi))\|_{L_p(U_1^+)} \\ &\le C\Big(C_p^2|Lu|_{W_p^{k-2m}(U_3^+)} \\ &\quad + (C_p + d^{-1})\sum_{j=1}^{m}\|v_j\|_{W_p^{k-m_j}(U_3^+)} \\ &\quad + C_p^2 d|u|_{W_p^k(U_3^+)} \\ &\quad + C_p d^{-1}(C_p + d^{-1})\|u\|_{W_p^{k-1}(U_3^+)}\Big).\end{aligned} \tag{2.96}$$

The lemma follows by summing Equation 2.96 over all $|\gamma| = k$. □

Next we consider the general case of a curved boundary. The idea is to flatten the boundary and use Lemma 2.9.

Lemma 2.10. *Suppose that $x_0 \in \partial\Omega$ and $0 < d \le d'$. Assume that d' and d/d' are sufficiently small. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{d'}(x_0) \cap \Omega$, and $T = B_{d'}(x_0) \cap \partial\Omega$. If $1 < p < \infty$, $k \ge k_0$, $u \in W_p^k(V^+)$, and, for each $j \in 1 : m$, $v_j \in W_p^{k-m_j}(V^+)$ is* *such that $B_j u = v_j$ on T, then*

$$
\begin{aligned}
|u|_{W_p^k(U^+)} \le C\Big(& C_p^2 |Lu|_{W_p^{k-2m}(V^+)} \\
&+ (C_p + d^{-1}) \sum_{j=1}^{m} \|v_j\|_{W_p^{k-m_j}(V^+)} \\
&+ C_p^2 d |u|_{W_p^k(V^+)} \\
&+ C_p d^{-1}(C_p + d^{-1}) \|u\|_{W_p^{k-1}(V^+)}\Big).
\end{aligned}
\tag{2.97}
$$

Proof. Let $U = B_d(x_0)$ and $V = B_{d'}(x_0)$.

For sufficiently small d', there exists an invertible and sufficiently smooth $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ which flattens the boundary of Ω in V and has sufficiently smooth inverse. With $\hat{x}_0 = \Phi(x_0)$, $\hat{U}^+ = \Phi(U^+)$, $\hat{V}^+ = \Phi(V^+)$, $\hat{T} = \Phi(T)$, $\hat{U} = \Phi(U)$, and $\hat{V} = \Phi(V)$, this means that $\hat{V}^+ \subset \mathbb{R}^N_+$ and $\hat{T} \subset \partial\mathbb{R}^N_+$. If d/d' is sufficiently small, there exists some $\hat{d} > 0$ such that $\hat{U} \subset B_{\hat{d}}(\hat{x}_0)$ and $B_{3\hat{d}}(\hat{x}_0) \subset \hat{V}$. Define the transformed operators $\hat{L}$ and $\hat{B}_j$ as in Section 2.4. Let $\hat{u} = u \circ \Phi^{-1}$ and $\hat{v}_j = v_j \circ \Phi^{-1}$.

Applying Lemma 2.9 to the transformed setup,

$$
\begin{aligned}
|\hat{u}|_{W_p^k(\hat{U}^+)} \le C\Big(& C_p^2 |\hat{L}\hat{u}|_{W_p^{k-2m}(\hat{V}^+)} \\
&+ (C_p + \hat{d}^{-1}) \sum_{j=1}^{m} \|\hat{v}_j\|_{W_p^{k-m_j}(\hat{V}^+)} \\
&+ C_p^2 \hat{d} |\hat{u}|_{W_p^k(\hat{V}^+)} \\
&+ C_p \hat{d}^{-1}(C_p + \hat{d}^{-1}) \|\hat{u}\|_{W_p^{k-1}(\hat{V}^+)}\Big).
\end{aligned}
\tag{2.98}
$$

We bound the left side of Equation 2.97 by

$$
\begin{aligned}
|u|_{W_p^k(U^+)} &\le C \|\hat{u}\|_{W_p^k(\hat{U}^+)} \\
&\le C\Big(|\hat{u}|_{W_p^k(\hat{U}^+)} + \|\hat{u}\|_{W_p^{k-1}(\hat{U}^+)}\Big).
\end{aligned}
\tag{2.99}
$$

The first term is bounded by the right side of Equation 2.98 and the second term is bounded by the fourth term on the right side of Equation 2.97. It remains to show that each of the four terms on the right side of Equation 2.98 are bounded by the right side of Equation 2.97.

The first term on the right side of Equation 2.98 has the factor

$$\begin{aligned} |\hat{L}\hat{u}|_{W_p^{k-2m}(\hat{V}^+)} &\le C\|Lu\|_{W_p^{k-2m}(V^+)} \\ &\le C\Big(|Lu|_{W_p^{k-2m}(V^+)} + \|u\|_{W_p^{k-1}(V^+)}\Big), \end{aligned} \tag{2.100}$$

and is thus bounded by the first and fourth terms on the right side of Equation 2.97. Since $\hat{d}^{-1} \le Cd^{-1}$, the second and fourth terms on the right side of Equation 2.98 are bounded by the second and fourth terms on the right side of Equation 2.97, respectively. Since $\hat{d} \le Cd$, the third term on the right side of Equation 2.98 is bounded by the third and fourth terms on the right side of Equation 2.97. $\square$

2.6.2 A Special Case

Throughout this subsection, we assume that $m = 1$ and $m_1 \in 0 : 1$. That is, we consider the case of second-order equations with boundary conditions of order at most one. We improve the boundary estimates of Section 2.6.1 in this special case.

We proceed in three stages. First, we assume that we start with a flat boundary portion and a differential operator whose leading part, at a point, is the Laplacian.

Lemma 2.11. *Suppose that $x_0 \in \partial\Omega$ and $d > 0$. Assume that $L_{x_0} = \Delta$. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{3d}(x_0) \cap \Omega$, and $T = B_{3d}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$. If $1 < p < \infty$, $k \ge 2$, $u \in W_p^k(V^+)$, and $v_1 \in W_p^{k-m_1}(V^+)$ is such that $B_1 u = v_1$ on T then*

$$\begin{aligned} |u|_{W_p^k(U^+)} \le C\Big(&C_p|Lu|_{W_p^{k-2}(V^+)} \\ &+ (C_p + d^{-1})\|v_1\|_{W_p^{k-m_1}(V^+)} \\ &+ C_p d|u|_{W_p^k(V^+)} \\ &+ (C_p + d^{-1})\|u\|_{W_p^{k-1}(V^+)}\Big). \end{aligned} \tag{2.101}$$

Proof. For $i \in 1:3$, let $U_i = B_{id}(x_0)$ and $U_i^+ = B_{id}(x_0) \cap \mathbb{R}^N_+$.

Let $|\zeta| = k-2$ have $\zeta_N = 0$ and let $|\gamma| = 2$. By the Bramble-Hilbert lemma, there exists some constant $\chi \in \Pi^0(U_3)$ such that, if $i \in 0:1$ then

$$|D^\zeta u - \chi|_{W_p^i(U_3^+)} \le C d^{1-i} |D^\zeta u|_{W_p^1(U_3^+)}. \tag{2.102}$$

Let $\omega \in C_0^\infty(U_3)$ be such that $\omega = 1$ on U_2 and, for $i \in 0:2$,

$$|\omega|_{W_\infty^i(U_3)} \le C d^{-i}. \tag{2.103}$$

We start by finding a function $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ such that $\omega(D^\zeta u - \chi) - v$ is harmonic on $\mathbb{R}^N_+$ and $B_{1,x_0} v = 0$ on $\partial\mathbb{R}^N_+$. Define $f : \mathbb{R}^N \to \mathbb{R}$ by

$$f(x) = \begin{cases} \Delta(\omega(D^\zeta u - \chi))(x), & \text{if } x_N > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.104}$$

By the general Leibniz rule,

$$f = g + h, \tag{2.105}$$

where

$$g = \begin{cases} \omega\Big(LD^\zeta u - \sum_{|\alpha|=2} (a_\alpha - a_\alpha(x_0)) D^{\alpha+\zeta} u + \sum_{|\alpha|<2} a_\alpha D^{\alpha+\zeta} u\Big), & \text{if } x_N > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.106}$$

and

$$h = \begin{cases} \sum_{\substack{|\alpha|+|\beta|=2 \\ |\alpha|>0}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (D^\zeta u - \chi), & \text{if } x_N > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2.107}$$

and the $c_{1,\alpha,\beta}$ are constants that depend on N. By Equation 2.106,

$$\|g\|_{L_p(U_3)} \le C\Big(\|LD^\zeta u\|_{L_p(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)} + d|u|_{W_p^k(U_3^+)}\Big). \tag{2.108}$$

If $|\alpha|+|\beta|=2$ and $|\alpha|>0$ then, by Equations 2.103 and 2.102,

$$\begin{aligned}\|D^\alpha\omega D^\beta(D^\zeta u-\chi)\|_{L_p(U_3^+)} &\le |\omega|_{W_\infty^{|\alpha|}(U_3^+)}|D^\zeta u-\chi|_{W_p^{|\beta|}(U_3^+)}\\ &\le Cd^{-|\alpha|}d^{1-|\beta|}|u|_{W_p^{k-1}(U_3^+)}\\ &= Cd^{-1}|u|_{W_p^{k-1}(U_3^+)}.\end{aligned}\tag{2.109}$$

By Equations 2.107 and 2.109,

$$\|h\|_{L_p(U_3)}\le Cd^{-1}|u|_{W_p^{k-1}(U_3^+)}.\tag{2.110}$$

Define $\Gamma:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}$ by

$$\Gamma(x)=\begin{cases}\dfrac{1}{2\pi}\log|x|, & \text{if } N=2\\ -\dfrac{1}{N(N-2)\operatorname{meas}_N(B^N)}|x|^{-(N-2)}, & \text{otherwise.}\end{cases}\tag{2.111}$$

By [12, Equations 2.12 and 2.17], Γ is the fundamental solution corresponding to Δ. Obviously the first-order derivatives of Γ are homogeneous of degree $-(N-1)$. Notice that $h=h^*=0$ outside of $U_3\setminus U_2$ and that, if $x\in U_1^+$ and $y\in U_3\setminus U_2$ then $d\le|x-y|\le 4d$. Therefore, by Corollary 2.25, Part 1,

$$\|D^\gamma\Gamma\,\hat{*}\,f\|_{L_p(U_1^+)}\le C\Big(C_p\|g\|_{L_p(U_3)}+\|h\|_{L_p(U_3)}\Big)\tag{2.112}$$

and

$$\|D^\gamma\Gamma\,\hat{*}\,f^*\|_{L_p(U_1^+)}\le C\Big(C_p\|g^*\|_{L_p(U_3)}+\|h^*\|_{L_p(U_3)}\Big).\tag{2.113}$$

First we consider the case $m_1=0$. By definition of m_1, if $v:\bar{\mathbb{R}}_+^N\to\mathbb{R}$ then $B_{1,x_0}v=b_0v$ for some constant b_0. For $x,y\in\bar{\mathbb{R}}_+^N$ and $x\ne y$, let

$$G(x,y)=\Gamma(x-y)-\Gamma(x-y^*).\tag{2.114}$$

Define $v:\bar{\mathbb{R}}_+^N\to\mathbb{R}$ by

$$\begin{aligned}v(x)&=\int_{\mathbb{R}_+^N}G(x,y)f(y)\,dy\\ &=\int_{\mathbb{R}^N}\Gamma(x-y)\Big(f(y)-f^*(y)\Big)\,dy.\end{aligned}\tag{2.115}$$

That is,

$$v = \Gamma * (f - f^*). \tag{2.116}$$

If $x, y \in \mathbb{R}^N_+$ then $x \neq y^*$, so $\Delta\Gamma(x - y^*) = 0$. Therefore $\Delta v = \Delta(\Gamma * f) = f$ on $\mathbb{R}^N_+$.

Observe that $\Gamma(z^*) = \Gamma(z)$ for $z \in \mathbb{R}^N \smallsetminus \{0\}$. Therefore, if $x \in \partial\mathbb{R}^N_+$ and $y \in \mathbb{R}^N_+$ then

$$\begin{aligned} B_{1,x_0}G(x, y) &= b_0(\Gamma(x - y) - \Gamma(x - y^*)) \\ &= b_0(\Gamma(x - y) - \Gamma(x^* - y^*)) \\ &= 0. \end{aligned} \tag{2.117}$$

By Equations 2.115 and 2.117, we see that $B_{1,x_0}v = 0$ on $\partial\mathbb{R}^N_+$.

By [3, Chapter 5, Equation 26] and Equation 2.116,

$$D^\gamma v = D^\gamma\Gamma \hat{*} (f - f^*) + c_2(f - f^*), \tag{2.118}$$

where c_2 is a constant that depends on γ. By Equations 2.118, 2.112, 2.113, and 2.105,

$$\begin{aligned} \|D^\gamma v\|_{L_p(U_1^+)} \leq C\Big(&C_p\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)} \\ &+ C_p\|g^*\|_{L_p(U_3)} + \|h^*\|_{L_p(U_3)}\Big). \end{aligned} \tag{2.119}$$

Next we consider the case $m_1 = 1$. By definition of m_1, if $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ then $B_{1,x_0}v = \sum_{i=1}^N b_i D_i v$ for some constants b_i. By the complementing condition, we must have $b \neq 0$. Following [12, Equation 6.62], define $\Theta : \mathbb{R}^N \smallsetminus \{0\} \to \mathbb{R}$ by

$$\Theta(x) = 2|b_N| \int_0^\infty D_N\Gamma\big(x + \operatorname{sign}(b_N)tb\big)\, dt. \tag{2.120}$$

Using the fact that the second-order derivatives of Γ are homogeneous of degree $-N$, it is easily seen that the first-order derivatives of Θ are homogeneous of degree $-(N - 1)$. Again using Corollary 2.25, Part 1,

$$\|D^\gamma\Theta \hat{*} f^*\|_{L_p(U_1^+)} \leq C\Big(C_p\|g^*\|_{L_p(U_3)} + \|h^*\|_{L_p(U_3)}\Big). \tag{2.121}$$

For $x, y \in \bar{\mathbb{R}}^N_+$ and $x \neq y$, let

$$G(x,y) = \Gamma(x-y) - \Gamma(x-y^*) + \Theta(x-y^*). \tag{2.122}$$

Define $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ by

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^N_+} G(x,y) f(y)\, dy \\ &= \int_{\mathbb{R}^N} \Big(\Gamma(x-y)f(y) - \Gamma(x-y)f^*(y) + \Theta(x-y)f^*(y) \Big)\, dy. \end{aligned} \tag{2.123}$$

That is,

$$v = \Gamma * (f - f^*) + \Theta * f^*. \tag{2.124}$$

If $x, y \in \mathbb{R}^N_+$ then $x \neq y^*$, so $\Delta\Gamma(x-y^*) = 0$. Since $\Delta\Gamma = 0$ on $\mathbb{R}^N_+$, $D_N\Delta\Gamma = 0$ on $\mathbb{R}^N_+$. If $z \in \mathbb{R}^N_+$ and $t > 0$ then $z + \mathrm{sign}(b_N)tb \in \mathbb{R}^N_+$, so $\Delta D_N\Gamma\big(z + \mathrm{sign}(b_N)tb\big) = 0$. By Equation 2.120, we see that, if $x, y \in \mathbb{R}^N_+$ then $x - y^* \in \mathbb{R}^N_+$, so $\Delta\Theta(x-y^*) = 0$. Therefore $\Delta v = \Delta(\Gamma * f) = f$ on $\mathbb{R}^N_+$.

Fix $z \in \mathbb{R}^N_+$ and define $\theta : (-z_N/|b|, z_N/|b|) \to \mathbb{R}$ by

$$\begin{aligned} \theta(s) &= \Theta\big(z + \mathrm{sign}(b_N)sb\big) \\ &= 2|b_N| \int_0^\infty D_N\Gamma\big(z + \mathrm{sign}(b_N)(t+s)b\big)\, dt \\ &= 2|b_N| \int_s^\infty D_N\Gamma\big(z + \mathrm{sign}(b_N)tb\big)\, dt. \end{aligned} \tag{2.125}$$

Then

$$D\theta(s) = -2|b_N| D_N\Gamma\big(z + \mathrm{sign}(b_N)sb\big). \tag{2.126}$$

In particular,

$$\sum_{i=1}^N b_i D_i\Theta(z) = \mathrm{sign}(b_N) D\theta(0) = -2b_N D_N\Gamma(z). \tag{2.127}$$

Observe that, for $z \in \mathbb{R}^N \setminus \{0\}$,

$$D_i\Gamma(z^*) = \begin{cases} D_i\Gamma(z), & \text{if } i \in 1 : N-1 \\ -D_N\Gamma(z), & \text{if } i = N. \end{cases} \tag{2.128}$$

Now, if $x \in \partial\mathbb{R}^N_+$ and $y \in \mathbb{R}^N_+$ then, by Equations 2.128 and 2.127,

$$\begin{aligned}
B_{1,x_0}G(x,y) &= \sum_{i=1}^{N} b_i\Big(D_i\Gamma(x-y) - D_i\Gamma(x-y^*) + D_i\Theta(x-y^*)\Big) \\
&= \sum_{i=1}^{N} b_i\Big(D_i\Gamma(x-y) - D_i\Gamma(x^*-y^*) + D_i\Theta(x^*-y^*)\Big) \\
&= -2b_N D_N\Gamma(x-y) - 2b_N D_N\Gamma(x^*-y^*) \\
&= 0.
\end{aligned} \tag{2.129}$$

By Equations 2.123 and 2.129, we see that $B_{1,x_0}v = 0$ on $\partial\mathbb{R}^N_+$.

By [3, Chapter 5, Equation 26] and Equation 2.124,

$$D^\gamma v = D^\gamma\Gamma \,\hat{*}\, (f - f^*) + D^\gamma\Theta \,\hat{*}\, f^* + c_3 f + c_4 f^*, \tag{2.130}$$

where c_3 and c_4 are constants that depend on γ. By Equations 2.130, 2.112, 2.113, 2.121, and 2.105,

$$\begin{aligned}
\|D^\gamma v\|_{L_p(U_1^+)} \le C\Big(&C_p\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)} \\
&+ C_p\|g^*\|_{L_p(U_3)} + \|h^*\|_{L_p(U_3)}\Big).
\end{aligned} \tag{2.131}$$

Now we rejoin the cases $m_1 = 0$ and $m_1 = 1$. In both cases, we have constructed a function $v : \bar{\mathbb{R}}^N_+ \to \mathbb{R}$ such that $\omega(D^\zeta u - \chi) - v$ is harmonic on $\mathbb{R}^N_+$ and $B_{1,x_0}v = 0$ on $\partial\mathbb{R}^N_+$. Using Equations 2.119, 2.131, 2.108, and 2.110, and summing over all $|\gamma| = 2$, we have that

$$\begin{aligned}
|v|_{W_p^2(U_1^+)} &\le C\Big(C_p\|g\|_{L_p(U_3)} + \|h\|_{L_p(U_3)}\Big) \\
&\le C\Big(C_p\|LD^\zeta u\|_{L_p(U_3^+)} \\
&\qquad + C_p d|u|_{W_p^k(U_3^+)} \\
&\qquad + (C_p + d^{-1})\|u\|_{W_p^{k-1}(U_3^+)}\Big).
\end{aligned} \tag{2.132}$$

Define $v_{1,\zeta} : U_3^+ \to \mathbb{R}$ by

$$v_{1,\zeta} = D^\zeta(v_1 - B_1 u) + B_1 D^\zeta u. \tag{2.133}$$

On T, $D^\zeta(v_1 - B_1 u)$ is a tangential derivative of a function which is zero on T, and is thus itself zero on T. Therefore, $B_1 D^\zeta u = v_{1,\zeta}$ on T.

Once again let $|\gamma| = 2$. We can now follow the proof of Lemma 2.9, starting from Equation 2.81, with all occurrences of u, v_1, and k replaced by $D^\zeta u$, $v_{1,\zeta}$, and 2, respectively.

If $|\eta| = 2 - m_j - 1$ and $\eta_N = 0$ then, on $\partial\mathbb{R}^N_+$, $D^\eta B_{j,x_0} v$ is a tangential derivative of a function which is zero on $\partial\mathbb{R}^N_+$, and is thus itself zero on $\partial\mathbb{R}^N_+$. That is, the last term on the right side of Equation 2.84 vanishes. If we omit the last term on the right side of Equation 2.85 then Equation 2.86 still holds. The last terms on the right sides of Equations 2.88, 2.91, 2.94, and 2.95 disappear, With the appropriate substitutions, Equation 2.95 reads

$$\begin{aligned} \|D^\gamma(\omega(D^\zeta u - \chi))\|_{L_p(U_1^+)} \leq C\Big(&|v|_{W_p^2(U_1^+)} \\ &+ (C_p + d^{-1})\|v_{1,\zeta}\|_{W_p^{2-m_1}(U_3^+)} \\ &+ C_p d |D^\zeta u|_{W_p^2(U_3^+)} \\ &+ (C_p + d^{-1})\|D^\zeta u\|_{W_p^1(U_3^+)}\Big). \end{aligned} \tag{2.134}$$

By the general Leibniz rule,

$$v_{1,\zeta} = D^\zeta v_1 - \sum_{|\beta| < k-2+m_1} c_{5,\beta} D^\beta u, \tag{2.135}$$

where the $c_{5,\beta} : \bar{U}_3^+ \to \mathbb{R}$ can be expressed in terms of ζ and the coefficients of B_1. Therefore,

$$\|v_{1,\zeta}\|_{W_p^{2-m_1}(U_3^+)} \leq C\Big(\|D^\zeta v_1\|_{W_p^{2-m_1}(U_3^+)} + \|u\|_{W_p^{k-1}(U_3^+)}\Big). \tag{2.136}$$

By Equations 2.134, 2.132, and 2.136, along with Proposition 2.6, we have that

$$
\begin{aligned}
\|D^\gamma D^\zeta u\|_{L_p(U_1^+)} &= \|D^\gamma(\omega(D^\zeta u - \chi))\|_{L_p(U_1^+)} \\
&\le C\Big(C_p|Lu|_{W_p^{k-2}(U_3^+)} \\
&\qquad + (C_p + d^{-1})\|v_1\|_{W_p^{k-m_1}(U_3^+)} \\
&\qquad + C_p d|u|_{W_p^k(U_3^+)} \\
&\qquad + (C_p + d^{-1})\|u\|_{W_p^{k-1}(U_3^+)}\Big).
\end{aligned}
\tag{2.137}
$$

Summing this inequality over all $|\gamma| = 2$, we see that $|D^\zeta u|_{W_p^2(U_1^+)}$ is bounded by the right side of Equation 2.137. The lemma follows by Proposition 2.7. □

Second, we assume that we start with a flat boundary but allow the differential operator to be arbitrary. The idea is to transform the domain, while preserving the flat boundary, so that the leading part of the transformed differential operator, at a certain point, is the Laplacian. We then use Lemma 2.11.

Lemma 2.12. *Suppose that $x_0 \in \partial\Omega$ and $0 < d \le d'$. Assume that d/d' is sufficiently small. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{d'}(x_0) \cap \Omega$, and $T = B_{d'}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$. If $1 < p < \infty$, $k \ge 2$, $u \in W_p^k(V^+)$, and $v_1 \in W_p^{k-m_1}(V^+)$ is such that $B_1 u = v_1$ on T then*

$$
\begin{aligned}
|u|_{W_p^k(U^+)} &\le C\Big(C_p|Lu|_{W_p^{k-2}(V^+)} \\
&\qquad + (C_p + d^{-1})\|v_1\|_{W_p^{k-m_1}(V^+)} \\
&\qquad + C_p d|u|_{W_p^k(V^+)} \\
&\qquad + (C_p + d^{-1})\|u\|_{W_p^{k-1}(V^+)}\Big).
\end{aligned}
\tag{2.138}
$$

Proof. Let $U = B_d(x_0)$ and $V = B_{d'}(x_0)$.

Define $A \in \mathbb{R}^{N\times N}$ by

$$
A_{i,j} = \frac{(e_i + e_j)!}{2!} a_{e_i+e_j}(x_0). \tag{2.139}
$$

Obviously A is symmetric. It is positive-definite because, if $\xi \in \mathbb{R}^N$ then, by uniform ellipticity,

$$\begin{aligned} \xi^\mathsf{T} A \xi &= \sum_{i,j=1}^{N} \frac{(e_i + e_j)!}{2!} a_{e_i+e_j}(x_0)\xi_i \xi_j \\ &= \sum_{|\alpha|=2} a_\alpha(x_0)\xi^\alpha \\ &\geq C|\xi|^2. \end{aligned} \tag{2.140}$$

Therefore, by the spectral theorem, there exists an orthogonal $Q \in \mathbb{R}^{N\times N}$ such that $QAQ^\mathsf{T} = D$, where $D \in \mathbb{R}^{N\times N}$ is the diagonal matrix of eigenvalues of A, all of which are positive.

Let $\xi = D^{1/2}Qe_N$ and let $P \in \mathbb{R}^{N\times N}$ be an orthogonal matrix whose Nth row is $\xi^\mathsf{T}/|\xi|$. Finally, define $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi(x) = PD^{-1/2}Qx$. Obviously $\Phi \in (C^\infty(\mathbb{R}^N))^N$ and has inverse $\Phi^{-1} \in (C^\infty(\mathbb{R}^N))^N$. Let $R = PD^{-1/2}Q$. Notice that $D\Phi(x) = R$ at every point $x \in \mathbb{R}^N$ and

$$\begin{aligned} RAR^\mathsf{T} &= (PD^{-1/2}Q)A(PD^{-1/2}Q)^\mathsf{T} \\ &= PD^{-1/2}(QAQ^\mathsf{T})D^{-1/2}P^\mathsf{T} \\ &= I. \end{aligned} \tag{2.141}$$

Let $\hat{x}_0 = \Phi(x_0)$, $\hat{U}^+ = \Phi(U^+)$, $\hat{V}^+ = \Phi(V^+)$, $\hat{T} = \Phi(T)$, $\hat{U} = \Phi(U)$, and $\hat{V} = \Phi(V)$. If d/d' is sufficiently small, there exists some $\hat{d} > 0$ such that $\hat{U} \subset B_{\hat{d}}(\hat{x}_0)$ and $B_{3\hat{d}}(\hat{x}_0) \subset \hat{V}$. Define the transformed operators $\hat{L}$ and $\hat{B}_j$ as in Section 2.4. Let $\hat{u} = u \circ \Phi^{-1}$ and $\hat{v}_j = v_j \circ \Phi^{-1}$.

First we show that the flat boundary is preserved. If $i \in 1 : N$ then

$$\begin{aligned} e_N^\mathsf{T} \Phi(e_i) &= e_N^\mathsf{T} P D^{-1/2} Q e_i \\ &= \frac{1}{|\xi|} \xi^\mathsf{T} D^{-1/2} Q e_i \\ &= \frac{1}{|\xi|} e_N^\mathsf{T} Q^\mathsf{T} D^{1/2} D^{-1/2} Q e_i \\ &= \frac{1}{|\xi|} e_N^\mathsf{T} e_i \\ &= \frac{1}{|\xi|} \delta_{i,N}. \end{aligned} \tag{2.142}$$

Since Φ is linear, this means that, if $x \in \mathbb{R}^N$ then

$$(\Phi(x))_N = e_N^\mathsf{T} \Phi(x) = \frac{1}{|\xi|} x_N. \tag{2.143}$$

From this we can conclude that $\hat{V}^+ \subset \mathbb{R}^N_+$ and $\hat{T} \subset \partial \mathbb{R}^N_+$.

Second we show that, at $\hat{x}_0$, the leading part of the transformed differential operator is the Laplacian. If $\hat{\xi} \in \mathbb{R}^N$ then, by Equations 2.35 and 2.141,

$$\begin{aligned} \sum_{|\alpha|=2} \hat{a}_\alpha(\hat{x}_0) \hat{\xi}^\alpha &= \sum_{|\alpha|=2} a_\alpha(x_0) \Big((D\Phi(x_0))^\mathsf{T} \hat{\xi} \Big)^\alpha \\ &= \sum_{i,j=1}^{N} A_{i,j} (R^\mathsf{T} \hat{\xi})_i (R^\mathsf{T} \hat{\xi})_j \\ &= \hat{\xi}^\mathsf{T} R A R^\mathsf{T} \hat{\xi} \\ &= |\hat{\xi}|^2. \end{aligned} \tag{2.144}$$

Taking $\hat{\xi} = e_i$ for some $i \in 1 : N$, we find that $\hat{a}_{e_i+e_i}(\hat{x}_0) = 1$. Taking $\hat{\xi} = e_i + e_j$ for some $i, j \in 1 : N$ with $i \neq j$, we find that $\hat{a}_{e_i+e_j}(\hat{x}_0) = 0$. This means that $\hat{L}_{\hat{x}_0} = \Delta$.

Applying Lemma 2.11 to the transformed setup,

$$\begin{aligned} |\hat{u}|_{W_p^k(\hat{U}^+)} \leq C \Big(& C_p |\hat{L}\hat{u}|_{W_p^{k-2}(\hat{V}^+)} \\ & + (C_p + \hat{d}^{-1}) \|\hat{v}_1\|_{W_p^{k-m_1}(\hat{V}^+)} \\ & + C_p \hat{d} |\hat{u}|_{W_p^k(\hat{V}^+)} \\ & + (C_p + \hat{d}^{-1}) \|\hat{u}\|_{W_p^{k-1}(\hat{V}^+)} \Big). \end{aligned} \tag{2.145}$$

The lemma follows from this in the same way that Lemma 2.10 follows from Equation 2.98. $\square$

Third, we consider the general case of a curved boundary and an arbitrary differential operator. The idea is to flatten the boundary and use Lemma 2.12.

Lemma 2.13. *Suppose that $x_0 \in \partial\Omega$ and $0 < d \le d'$. Assume that d' and d/d' are sufficiently small. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{d'}(x_0) \cap \Omega$, and $T = B_{d'}(x_0) \cap \partial\Omega$. If $1 < p < \infty$, $k \ge k_0$, $u \in W_p^k(V^+)$, and $v_1 \in W_p^{k-m_1}(V^+)$ is such that $B_1 u = v_1$ on T then*

$$\begin{aligned} |u|_{W_p^k(U^+)} \le C\Big(&C_p |Lu|_{W_p^{k-2}(V^+)} \\ &+ (C_p + d^{-1})\|v_1\|_{W_p^{k-m_1}(V^+)} \\ &+ C_p d |u|_{W_p^k(V^+)} \\ &+ (C_p + d^{-1})\|u\|_{W_p^{k-1}(V^+)}\Big). \end{aligned} \tag{2.146}$$

Proof. Let $U = B_d(x_0)$ and $V = B_{d'}(x_0)$.

We proceed with a slight modification of the proof of Lemma 2.10. For sufficiently small d', there exists an invertible and sufficiently smooth $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ which flattens the boundary of Ω in V and has sufficiently smooth inverse. With $\hat{x}_0 = \Phi(x_0)$, $\hat{U}^+ = \Phi(U^+)$, $\hat{V}^+ = \Phi(V^+)$, $\hat{T} = \Phi(T)$, $\hat{U} = \Phi(U)$, and $\hat{V} = \Phi(V)$, this means that $\hat{V}^+ \subset \mathbb{R}^N_+$ and $\hat{T} \subset \partial\mathbb{R}^N_+$. Given $\epsilon > 0$, if d/d' is sufficiently small, there exist $\hat{d}, \hat{d}' > 0$ with $\hat{d}/\hat{d}' \le \epsilon$ such that $\hat{U} \subset B_{\hat{d}}(\hat{x}_0)$ and $B_{\hat{d}'}(\hat{x}_0) \subset \hat{V}$. The lemma follows from Lemma 2.12 in the same way that Lemma 2.10 follows from Lemma 2.9. $\square$

2.7 Green's Function Estimates

In this section, we prove Theorem 2.5.

By the uniqueness assumption, we have by [17, Corollary to Theorem 3.3] that, for $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y) Lu(y)\, dy, \tag{2.147}$$

where the derivatives of the Green's function G satisfy the estimates $|D^\alpha G(x,y)| \leq C\bar{G}_k(x-y)$ for $|\alpha| = k$, where

$$|\bar{G}_k(z)| \leq \begin{cases} 1 + |\log \frac{1}{|z|}|, & \text{if } 2m - N - k = 0 \\ |z|^{2-N-k}, & \text{otherwise.} \end{cases} \tag{2.148}$$

Let $W = \{x - y : x \in U, y \in V\}$. If $2m - N - k = 0$ then

$$\|\bar{G}_k\|_{L_1(W)} \leq C \int_{d_1}^{d_2} \Big(1 + |\log \frac{1}{r}|\Big) r^{N-1}\, dr \leq C d_2^N \Big(1 + |\log \frac{1}{d_2}|\Big). \tag{2.149}$$

If $2m - k > 0$ but $2m - k \neq N$ then

$$\|\bar{G}_k\|_{L_1(W)} \leq C \int_{d_1}^{d_2} r^{2m-N-k} r^{N-1}\, dr \leq C d_2^{2m-k}. \tag{2.150}$$

If $2m - k = 0$ and $d_1 > 0$ then

$$\|\bar{G}_k\|_{L_1(W)} \leq C \int_{d_1}^{d_2} r^{2m-N-k} r^{N-1}\, dr \leq C \log \frac{d_2}{d_1}. \tag{2.151}$$

If $2m - k < 0$ and $d_1 > 0$ then

$$\|\bar{G}_k\|_{L_1(W)} \leq C \int_{d_1}^{d_2} r^{2m-N-k} r^{N-1}\, dr \leq C d_1^{-(k-2m)}. \tag{2.152}$$

Young's inequality, along with Equations 2.149, 2.150, 2.151, and 2.152, prove Parts 1, 2, 3, and 4, respectively.

2.8 Global Estimates

2.8.1 The General Case

In this subsection, we prove Theorem 2.1.

Let $d > 0$, $d_{\text{int}} = d$, $d'_{\text{int}} = 3d_{\text{int}}$, $d_{\text{bdry}} = d'_{\text{int}}$, $d'_{\text{bdry}} \geq d_{\text{bdry}}$, and define $\Omega_{\text{int}} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq d'_{\text{int}}\}$. For $x \in \Omega_{\text{int}}$, let $U_x = B_{d_{\text{int}}}(x)$ and $V_x = B_{d'_{\text{int}}}(x)$, and notice that $V_x \subset \Omega$. For $x \in \partial\Omega$, let $U_x = B_{d_{\text{bdry}}}(x) \cap \Omega$ and $V_x = B_{d'_{\text{bdry}}}(x) \cap \Omega$.

There exist finite subsets X_1 and X_2 of Ω_{int} and $\partial\Omega$, respectively, such that Ω is covered by the open sets U_x for $x \in X_1 \cup X_2$ and no point of Ω is in more than C of the sets V_x for $x \in X_1 \cup X_2$. The sizes of the sets X_1 and X_2 are irrelevant.

If $x \in X_1$ then, using Lemma 2.8 and the fact that $|\cdot|_1$ and $|\cdot|_p$ are equivalent on the finite-dimensional vector space $\mathbb{R}^3$,

$$\begin{aligned} |u|^p_{W^k_p(U_x)} \leq C^p \Big(& C_p^p |Lu|^p_{W^{k-2m}_p(V_x)} \\ & + (C_p d)^p |u|^p_{W^k_p(V_x)} \\ & + (C_p + d^{-1})^p \|u\|^p_{W^{k-1}_p(V_x)} \Big). \end{aligned} \tag{2.153}$$

If $d_{\text{bdry}}/d'_{\text{bdry}}$ is sufficiently small and $x \in X_2$ then, using Lemma 2.10 and the fact that $|\cdot|_1$ and $|\cdot|_p$ are equivalent on the finite-dimensional vector space $\mathbb{R}^{m+3}$,

$$\begin{aligned} |u|^p_{W^k_p(U_x)} \leq C^p \Big(& (C_p^2)^p |Lu|^p_{W^{k-2m}_p(V_x)} \\ & + (C_p + d^{-1})^p \sum_{j=1}^m \|v_j\|^p_{W^{k-m_j}_p(V_x)} \\ & + (C_p^2 d)^p |u|^p_{W^k_p(V_x)} \\ & + (C_p d^{-1})^p (C_p + d^{-1})^p \|u\|^p_{W^{k-1}_p(V_x)} \Big). \end{aligned} \tag{2.154}$$

By Equations 2.153 and 2.154,

$$\begin{aligned} |u|^p_{W^k_p(\Omega)} &\leq \sum_{x \in X_1 \cup X_2} |u|^p_{W^k_p(U_x)} \\ &\leq C^p \Big((C_p^2)^p |Lu|^p_{W^{k-2m}_p(\Omega)} \\ &\qquad + (C_p + d^{-1})^p \sum_{j=1}^m \|v_j\|^p_{W^{k-m_j}_p(\Omega)} \\ &\qquad + (C_p^2 d)^p |u|^p_{W^k_p(\Omega)} \\ &\qquad + (C_p d^{-1})^p (C_p + d^{-1})^p \|u\|^p_{W^{k-1}_p(\Omega)} \Big). \end{aligned} \tag{2.155}$$

Taking pth roots of both sides of this equation and using the fact that $|\cdot|_p$ and $|\cdot|_1$ are equivalent on the finite-dimensional vector space $\mathbb{R}^{m+3}$, we find that

$$\begin{aligned} |u|_{W_p^k(\Omega)} \leq C\Big(& C_p^2 |Lu|_{W_p^{k-2m}(\Omega)} \\ & + (C_p + d^{-1}) \sum_{j=1}^{m} \|v_j\|_{W_p^{k-m_j}(\Omega)} \\ & + C_p^2 d |u|_{W_p^k(\Omega)} \\ & + C_p d^{-1}(C_p + d^{-1}) \|u\|_{W_p^{k-1}(\Omega)} \Big). \end{aligned} \tag{2.156}$$

Choosing d so that $d \leq \frac{1}{2} C C_p^2$ and kicking back the second-last term on the right side, we obtain the theorem.

It is possible to iterate this result and use Sobolev inequalities to replace the $\|u\|_{W_p^{k-1}(\Omega)}$ on the right side of the estimate of Theorem 2.1 with even lower order norms of u. However, this will require additional factors of C_p to be multiplied by the other factors on the right side. We will not explore this option here.

2.8.2 A Special Case

Throughout this subsection, we assume that $m = 1$ and $m_1 \in 0 : 1$. That is, we consider the case of second-order equations with boundary conditions of order at most one. We use the results of Section 2.6.2 to improve the global estimates of Theorem 2.1 in this special case.

First we prove Theorem 2.2. We proceed as in Section 2.8.1 to obtain Equation 2.153. This time, if $d_{\mathrm{bdry}}/d'_{\mathrm{bdry}}$ is sufficiently small and $x \in X_2$ then, using Lemma 2.13 and the fact that $|\cdot|_1$ and $|\cdot|_p$ are equivalent on the finite-dimensional vector space $\mathbb{R}^4$,

$$\begin{aligned}|u|^p_{W^k_p(U_x)} \le C^p\Big(&C^p_p|Lu|^p_{W^{k-2}_p(V_x)}\\&+(C_p+d^{-1})^p\|v_1\|^p_{W^{k-m_1}_p(V_x)}\Big)\\&+(C_pd)^p|u|^p_{W^k_p(V_x)}\\&+(C_p+d^{-1})^p\|u\|^p_{W^{k-1}_p(V_x)}\Big).\end{aligned} \tag{2.157}$$

The theorem follows from Equations 2.153 and 2.157 in the same way that Theorem 2.1 follows from Equations 2.153 and 2.154.

As remarked at the end of the proof of Theorem 2.1 in Section 2.8.1, it is possible to use Sobolev inequalities to replace the $\|u\|_{W^{k-1}_p(\Omega)}$ on the right side of the estimate of Theorem 2.2 with even lower order norms of u. Doing so in this situation does not require additional factors of C_p to be multiplied by the other factors on the right side. In fact, if we make an assumption about the unique solvability of our boundary-value problem, we can dispose of the norm of u on the right side entirely without introducing extra factors of C_p. This is done in Theorem 2.3, which we now prove. We consider only the case of homogeneous boundary conditions.

If $x, y \in \Omega$ then obviously $|x-y| \le C$, so, by Theorem 2.5, Parts 1 and 2,

$$\|u\|_{W^1_p(\Omega)} \le C\|Lu\|_{L_p(\Omega)}. \tag{2.158}$$

Therefore, in order to prove the theorem, it remains only to show that

$$\|u\|_{W^k_p(\Omega)} \le CC_p\Big(\|Lu\|_{W^{k-2}_p(\Omega)} + \|u\|_{W^1_p(\Omega)}\Big). \tag{2.159}$$

First, we consider the case $k=2$. By Theorem 2.2,

$$\begin{aligned}\|u\|_{W^2_p(\Omega)} &\le |u|_{W^2_p(\Omega)} + \|u\|_{W^1_p(\Omega)}\\&\le CC_p\Big(\|Lu\|_{L_p(\Omega)} + \|u\|_{W^1_p(\Omega)}\Big).\end{aligned} \tag{2.160}$$

Second, we consider the case $k \geq 3$ and $p \leq \frac{N}{N-1}$. By the measure inequality, [2, Theorem 15.2], a Sobolev inequality, and the measure inequality once more,

$$\begin{aligned}
\|u\|_{W_p^{k-1}(\Omega)} &\leq C\|u\|_{W^{k-1}_{\frac{N}{N-1}}(\Omega)} \\
&\leq C\Big(\|Lu\|_{W^{k-3}_{\frac{N}{N-1}}(\Omega)} + \|u\|_{L_{\frac{N}{N-1}}(\Omega)}\Big) \\
&\leq C\Big(\|Lu\|_{W_1^{k-2}(\Omega)} + \|u\|_{W_1^1(\Omega)}\Big) \\
&\leq C\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{W_p^1(\Omega)}\Big).
\end{aligned} \tag{2.161}$$

Therefore, by Theorem 2.2,

$$\begin{aligned}
\|u\|_{W_p^k(\Omega)} &\leq |u|_{W_p^k(\Omega)} + \|u\|_{W_p^{k-1}(\Omega)} \\
&\leq CC_p\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{W_p^{k-1}(\Omega)}\Big) \\
&\leq CC_p\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{W_p^1(\Omega)}\Big).
\end{aligned} \tag{2.162}$$

Third, we consider the case $p \geq 2N$. By the measure inequality, a Sobolev inequality, [2, Theorem 15.2], and the measure inequality once more,

$$\begin{aligned}
\|u\|_{W_p^{k-1}(\Omega)} &\leq \|u\|_{W_\infty^{k-1}(\Omega)} \\
&\leq C\|u\|_{W_{2N}^k(\Omega)} \\
&\leq C\Big(\|Lu\|_{W_{2N}^{k-2}(\Omega)} + \|u\|_{L_{2N}(\Omega)}\Big) \\
&\leq C\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{L_p(\Omega)}\Big).
\end{aligned} \tag{2.163}$$

Therefore, by Theorem 2.2,

$$\begin{aligned}
\|u\|_{W_p^k(\Omega)} &\leq |u|_{W_p^k(\Omega)} + \|u\|_{W_p^{k-1}(\Omega)} \\
&\leq CC_p\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{W_p^{k-1}(\Omega)}\Big) \\
&\leq CC_p\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{L_p(\Omega)}\Big).
\end{aligned} \tag{2.164}$$

Fourth, we consider the case $k \geq 3$ and $\frac{N}{N-1} \leq p \leq 2N$. By [2, Theorem 15.2],

$$\|u\|_{W_p^k(\Omega)} \leq C\Big(\|Lu\|_{W_p^{k-2}(\Omega)} + \|u\|_{L_p(\Omega)}\Big). \tag{2.165}$$

2.9 Local Estimates

2.9.1 Domains in the Interior

Lemma 2.14. *Suppose that $x_0 \in \Omega$ and $d > 0$. Let $U = B_d(x_0)$ and $V = B_{2d}(x_0)$ and assume that $V \subset \Omega$. If $\frac{N}{N-1} \leq p \leq 2N$, $k \geq k_0$, $u \in W_p^k(V)$, and $Lu = 0$ on V then*

$$|u|_{W_p^k(U)} \leq Cd^{-1}\|u\|_{W_p^{k-1}(V)}. \tag{2.166}$$

Proof. By the Bramble-Hilbert lemma, there exists some $\chi \in \Pi^{k-2}(V)$ such that, if $i \in 0 : k-1$ then

$$|u - \chi|_{W_p^i(V)} \leq Cd^{k-1-i}|u|_{W_p^{k-1}(V)}. \tag{2.167}$$

Let $\omega \in C_0^\infty(V)$ be such that $\omega = 1$ on U and, for $i \in 0 : k$,

$$|\omega|_{W_\infty^i(V)} \leq Cd^{-i}. \tag{2.168}$$

Since $B_j(\omega(u - \chi)) = 0$ on $\partial\Omega$ for all $j \in 1 : m$, we have by Theorem 2.1 that

$$\begin{aligned} |u|_{W_p^k(U)} &= |\omega(u - \chi)|_{W_p^k(U)} \\ &\leq C\Big(|L(\omega(u - \chi))|_{W_p^{k-2m}(V)} + \|\omega(u - \chi)\|_{W_p^{k-1}(V)}\Big). \end{aligned} \tag{2.169}$$

Here we have used the fact that $C_p \leq 2N$.

Let $|\gamma| = k - 2m$. By the general Leibniz rule,

$$D^\gamma L(\omega(u - \chi)) = -\omega D^\gamma \chi + \sum_{\substack{|\alpha|+|\beta|\leq k \\ |\beta|<k}} c_{\alpha,\beta} D^\alpha \omega D^\beta (u - \chi), \tag{2.170}$$

where the $c_{\alpha,\beta} : V \to \mathbb{R}$ can be expressed in terms of γ and the coefficients of L. Using the fact that χ is a polynomial of degree at most $k-2$, along with Equation

2.167,

$$
\begin{aligned}
\|D^\gamma L\chi\|_{L_p(V)} &\le C\|\chi\|_{W_p^k(V)} \\
&= C\|\chi\|_{W_p^{k-1}(V)} \\
&\le C\Big(\|u-\chi\|_{W_p^{k-1}(V)} + \|u\|_{W_p^{k-1}(V)}\Big) \\
&\le C\|u\|_{W_p^{k-1}(V)}.
\end{aligned}
\tag{2.171}
$$

If $|\alpha| + |\beta| \le k$ and $|\beta| < k$ then, by Equations 2.168 and 2.167,

$$
\begin{aligned}
\|D^\alpha \omega D^\beta (u-\chi)\|_{L_p(V)} &\le |\omega|_{W_\infty^{|\alpha|}(V)} \|u-\chi\|_{W_p^{|\beta|}(V)} \\
&\le C d^{-|\alpha|} d^{(k-1)-|\beta|} |u|_{W_p^{k-1}(V)} \\
&= C d^{-1} |u|_{W_p^{k-1}(V)}.
\end{aligned}
\tag{2.172}
$$

Using Equations 2.170, 2.171, and 2.172 and summing over all $|\gamma| = k - 2m$, we can estimate the first term on the right side of Equation 2.169,

$$
|L(\omega(u-\chi))|_{W_p^{k-2m}(V)} \le C d^{-1} \|u\|_{W_p^{k-1}(V)}. \tag{2.173}
$$

The second term on the right side of Equation 2.169 is estimated by Equations 2.168 and 2.167,

$$
\begin{aligned}
\|\omega(u-\chi)\|_{W_p^{k-1}(V)} &\le C \sum_{i=0}^{k-1} \|\omega\|_{W_\infty^i(V)} \|u-\chi\|_{W_p^{k-1-i}(V)} \\
&\le C \sum_{i=0}^{k-1} d^{-i} d^{(k-1)-(k-1-i)} |u|_{W_p^{k-1}(V)} \\
&\le C |u|_{W_p^{k-1}(V)}.
\end{aligned}
\tag{2.174}
$$

Equations 2.169, 2.173, and 2.174 prove the lemma. □

Lemma 2.15. *Suppose that $x_0 \in \Omega$ and $d > 0$. Let $U = B_d(x_0)$ and $V = B_{2d}(x_0)$ and assume that $V \subset \Omega$. If $k \ge k_0$, $u \in W_1^k(V)$, and $Lu = 0$ on V then*

$$
|u|_{W^k_{\frac{N}{N-1}}(U)} \le C d^{-1} \|u\|_{W_1^k(V)}. \tag{2.175}
$$

Proof. By the Bramble-Hilbert lemma, there exists some $\chi \in \Pi^{k-1}(V)$ such that, if $i \in 0 : k$ then

$$|u - \chi|_{W_1^i(V)} \leq Cd^{k-i}|u|_{W_1^k(V)}. \tag{2.176}$$

Let $\omega \in C_0^\infty(V)$ be such that $\omega = 1$ on U and, for $i \in 0 : k$,

$$|\omega|_{W_\infty^i(V)} \leq Cd^{-i}. \tag{2.177}$$

Since $B_j(\omega(u - \chi)) = 0$ on $\partial\Omega$ for all $j \in 1 : m$, we have by Theorem 2.1 that

$$\begin{aligned} |u|_{W^k_{\frac{N}{N-1}}(U)} &= |\omega(u-\chi)|_{W^k_{\frac{N}{N-1}}(U)} \\ &\leq C\Big(|L(\omega(u-\chi))|_{W^{k-2m}_{\frac{N}{N-1}}(V)} + \|\omega(u-\chi)\|_{W^{k-1}_{\frac{N}{N-1}}(V)}\Big). \end{aligned} \tag{2.178}$$

Let $|\gamma| = k - 2m$. By the general Leibniz rule,

$$D^\gamma L(\omega(u-\chi)) = -\omega D^\gamma \chi + \sum_{\substack{|\alpha|+|\beta|\leq k \\ |\beta|<k}} c_{\alpha,\beta} D^\alpha \omega D^\beta (u - \chi), \tag{2.179}$$

where the $c_{\alpha,\beta} : V \to \mathbb{R}$ can be expressed in terms of γ and the coefficients of L. Using the fact that χ is a polynomial of degree at most $k-1$, along with a scaled Sobolev inequality and Equation 2.176,

$$\begin{aligned} \|D^\gamma L\chi\|_{L_{\frac{N}{N-1}}(V)} &\leq C\|\chi\|_{W^k_{\frac{N}{N-1}}(V)} \\ &= Cd^{-1}\|\chi\|_{W_1^k(V)} \\ &\leq Cd^{-1}\Big(\|u-\chi\|_{W_1^k(V)} + \|u\|_{W_1^k(V)}\Big) \\ &\leq Cd^{-1}\|u\|_{W_1^k(V)}. \end{aligned} \tag{2.180}$$

If $|\alpha| + |\beta| \leq k$ and $|\beta| < k$ then, by a scaled Sobolev inequality and Equations

2.177 and 2.176,

$$\begin{aligned}\|D^\alpha\omega D^\beta(u-\chi)\|_{L_{\frac{N}{N-1}}(V)} &\le |\omega|_{W_\infty^{|\alpha|}(V)}|u-\chi|_{W^{|\beta|}_{\frac{N}{N-1}}(V)}\\ &\le Cd^{-|\alpha|}\Big(d^{-1}|u-\chi|_{W_1^{|\beta|}(V)}\\ &\qquad + |u-\chi|_{W_1^{|\beta|+1}(V)}\Big)\\ &\le Cd^{-|\alpha|}\Big(d^{-1}d^{k-|\beta|}|u|_{W_1^k(V)}\\ &\qquad + d^{k-(|\beta|+1)}|u|_{W_1^k(V)}\Big)\\ &\le Cd^{-1}|u|_{W_1^k(V)}.\end{aligned} \tag{2.181}$$

Using Equations 2.179, 2.180, and 2.181 and summing over all $|\gamma| = k-2m$, we can estimate the first term on the right side of Equation 2.178,

$$|L(\omega(u-\chi))|_{W^{k-2m}_{\frac{N}{N-1}}(V)} \le Cd^{-1}\|u\|_{W_1^k(V)}. \tag{2.182}$$

The second term on the right side of Equation 2.178 is estimated by a scaled Sobolev inequality and Equations 2.177 and 2.176,

$$\begin{aligned}\|\omega(u-\chi)\|_{W^{k-1}_{\frac{N}{N-1}}(V)} &\le C\sum_{i=0}^{k-1}|\omega|_{W_\infty^i(V)}\|u-\chi\|_{W^{k-1-i}_{\frac{N}{N-1}}(V)}\\ &\le C\sum_{i=0}^{k-1}d^{-i}\Big(d^{-1}\|u-\chi\|_{W_1^{k-1-i}(V)}\\ &\qquad + \|u-\chi\|_{W_1^{k-i}(V)}\Big)\\ &\le C\sum_{i=0}^{k-1}d^{-i}\Big(d^{-1}d^{k-(k-1-i)}|u|_{W_1^k(V)}\\ &\qquad + d^{k-(k-i)}|u|_{W_1^k(V)}\Big)\\ &\le C|u|_{W_1^k(V)}.\end{aligned} \tag{2.183}$$

Equations 2.178, 2.182, and 2.183 prove the lemma. □

Lemma 2.16. *Suppose that $x_0 \in \Omega$ and $d > 0$. Let $U = B_d(x_0)$ and $V = B_{2d}(x_0)$ and assume that $V \subset \Omega$. If $k \ge k_0$, $u \in W_{2N}^k(V)$, and $Lu = 0$ on V then*

$$|u|_{W_\infty^{k-1}(U)} \le Cd^{-1/2}\|u\|_{W_{2N}^{k-1}(V)}. \tag{2.184}$$

Proof. By a scaled Sobolev inequality,

$$|u|_{W^{k-1}_\infty(U)} \le Cd^{1/2}\Big(d^{-1}\|u\|_{W^{k-1}_{2N}(U)} + |u|_{W^k_{2N}(U)}\Big). \tag{2.185}$$

By Lemma 2.14,

$$|u|_{W^k_{2N}(U)} \le Cd^{-1}\|u\|_{W^{k-1}_{2N}(V)}. \tag{2.186}$$

Equations 2.185 and 2.186 prove the lemma. □

Lemma 2.17. *Suppose that $x_0 \in \Omega$, $d > 0$, and $k \ge k_0$. Let $U = B_d(x_0)$ and $V = B_{2^k d}(x_0)$ and assume that $V \subset \Omega$.*

1. *If $\frac{N}{N-1} \le p \le 2N$, $u \in W^k_p(V)$, $Lu = 0$ on V, and $\ell \in k_0 - 1 : k$ then*

$$\|u\|_{W^k_p(U)} \le Cd^{-(k-\ell)}\|u\|_{W^\ell_p(V)}. \tag{2.187}$$

2. *If $1 \le p \le \frac{N}{N-1}$, $u \in W^k_p(V)$, $Lu = 0$ on V, and $\ell \in k_0 : k$ then*

$$\|u\|_{W^k_p(U)} \le Cd^{-(k-\ell)}\|u\|_{W^\ell_p(V)}. \tag{2.188}$$

3. *If $2N \le p \le \infty$, $u \in W^k_{2N}(V)$, $Lu = 0$ on V, and $\ell \in k_0 : k$ then*

$$\|u\|_{W^{k-1}_p(U)} \le Cd^{-(k-\ell)}\|u\|_{W^{\ell-1}_p(V)}. \tag{2.189}$$

Proof. For $i \in 0 : k$, let $U_i = B_{2^i d}(x_0)$, so that $U = U_0$ and $V = U_k$. First consider Part 1. Iterating Lemma 2.14 $k - \ell$ times,

$$\|u\|_{W^k_p(U_0)} \le Cd^{-(k-\ell)}\|u\|_{W^\ell_p(U_{k-\ell})}, \tag{2.190}$$

which proves part 1.

Next consider part 2. By the measure inequality,

$$\|u\|_{W^k_p(U_0)} \le Cd^{1-N/p'}\|u\|_{W^k_{\frac{N}{N-1}}(U_0)}. \tag{2.191}$$

Iterating Lemma 2.14 $k - \ell$ times,

$$\|u\|_{W^k_{\frac{N}{N-1}}(U_0)} \leq Cd^{-(k-\ell)} \|u\|_{W^\ell_{\frac{N}{N-1}}(U_{k-\ell})}. \tag{2.192}$$

By Lemma 2.15,

$$\|u\|_{W^\ell_{\frac{N}{N-1}}(U_{k-\ell})} \leq Cd^{-1} \|u\|_{W^\ell_1(U_{k-\ell+1})}. \tag{2.193}$$

Again using the measure inequality,

$$\|u\|_{W^\ell_1(U_{k-\ell+1})} \leq Cd^{N/p'} \|u\|_{W^\ell_p(U_{k-\ell+1})}. \tag{2.194}$$

Putting together Equations 2.191, 2.192, 2.193, and 2.194 proves part 2.

Next consider part 3. By the measure inequality,

$$\|u\|_{W^{k-1}_p(U_0)} \leq Cd^{N/p} \|u\|_{W^{k-1}_\infty(U_0)}. \tag{2.195}$$

By Lemma 2.16,

$$\|u\|_{W^{k-1}_\infty(U_0)} \leq Cd^{-1/2} \|u\|_{W^{k-1}_{2N}(U_1)}. \tag{2.196}$$

Iterating Lemma 2.14 $k - \ell$ times,

$$\|u\|_{W^{k-1}_{2N}(U_1)} \leq Cd^{-(k-\ell)} \|u\|_{W^{\ell-1}_{2N}(U_{k-\ell+1})}. \tag{2.197}$$

Again using the measure inequality,

$$\|u\|_{W^{\ell-1}_{2N}(U_{k-\ell+1})} \leq Cd^{1/2-N/p} \|u\|_{W^{\ell-1}_p(U_{k-\ell+1})}. \tag{2.198}$$

Putting together Equations 2.195, 2.196, 2.197, and 2.198 proves part 3. □

2.9.2 Domains at the Boundary

Throughout this subsection, we assume that $m = 1$ and $m_1 \in 0 : 1$. That is, we consider the case of second-order equations with boundary conditions of order at most one.

Lemma 2.18. *Suppose that $x_0 \in \partial\Omega$ and $d > 0$. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{2d}(x_0) \cap \Omega$, and $T = B_{2d}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$. If $\frac{N}{N-1} \leq p \leq 2N$, $k \geq 2$, $u \in W^k_p(V^+)$, $Lu = 0$ on V^+, and $B_1 u = 0$ on T then*

$$|u|_{W^k_p(U^+)} \leq C d^{-1} \|u\|_{W^{k-1}_p(V^+)}. \tag{2.199}$$

Proof. Define $U = B_d(x_0)$ and $V = B_{2d}(x_0)$.

Let $\omega \in C^\infty_0(V)$ be such that $\omega = 1$ on U and, for $i \in 0 : k$,

$$|\omega|_{W^i_\infty(V)} \leq C d^{-i}. \tag{2.200}$$

Let $|\zeta| = k - 2$ have $\zeta_N = 0$. Our first goal is to show that there exist $\chi \in \Pi^0(V^+)$ and $v_1 \in W^{2-m_1}_p(\Omega)$ such that $B_1(\omega(D^\zeta u - \chi)) = v_1$ on $\partial\Omega$,

$$\|v_1\|_{W^{2-m_1}_p(V^+)} \leq C d^{-1} \|u\|_{W^{k-1}_p(V^+)}, \tag{2.201}$$

and, for $i \in 0 : 1$,

$$|D^\zeta u - \chi|_{W^i_p(V^+)} \leq C d^{1-i} |u|_{W^{k-1}_p(V^+)}. \tag{2.202}$$

First we consider the case $m_1 = 0$. Here we take $\chi = 0$ and $v_1 = 0$. By definition of m_1, if $v : \bar{\Omega} \to \mathbb{R}$ then $B_1 v = b_0 v$ for some $b_0 : \bar{\Omega} \to \mathbb{R}$. By the complementing condition, we must have $b_0(x) \neq 0$ for all $x \in \partial\Omega$. Therefore, $u = 0$ on $\partial\Omega$. On T, $D^\zeta u$ is a tangential derivative of a function which is zero on T, and is thus itself zero on T. Equation 2.202 follows by the mean value theorem. Also observe that, on T,

$$B_1(\omega(D^\zeta u - \chi)) = b_0 \omega(D^\zeta u - \chi) = 0 = v_1. \tag{2.203}$$

Equation 2.201 is obvious.

Next we consider the case $m_1 = 1$. By the Bramble-Hilbert lemma, there exists some $\eta \in \Pi^{k-2}(V^+)$ such that, if $i \in 0 : k - 1$ then

$$|u - \eta|_{W^i_p(V^+)} \leq C d^{k-1-i} |u|_{W^{k-1}_p(V^+)}. \tag{2.204}$$

By the general Leibniz rule,

$$B_1(\omega D^\zeta(u-\eta)) = \omega D^\zeta B_1(u-\eta) + \sum_{\substack{|\alpha|+|\beta|\le k-1 \\ |\alpha|\le 1}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (u-\eta), \tag{2.205}$$

where the $c_{1,\alpha,\beta} : \bar{\Omega} \to \mathbb{R}$ can be expressed in terms of ζ and the coefficients of B_1. On T, $D^\zeta B_1 u$ is a tangential derivative of a function which is zero on T, and is thus itself zero on T. Therefore, if we define

$$v_1 = -\omega D^\zeta B_1 \eta + \sum_{\substack{|\alpha|+|\beta|\le k-1 \\ |\alpha|\le 1}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (u-\eta), \tag{2.206}$$

then, by Equation 2.205, $B_1(\omega D^\zeta(u-\eta)) = v_1$ on $\partial\Omega$. Observe that

$$\begin{aligned} \|v_1\|_{W_p^1(V^+)} \le C\Big(&\|\omega D^\zeta B_1 \eta\|_{W_p^1(V^+)} \\ &+ \sum_{\substack{|\alpha|+|\beta|\le k \\ |\alpha|\le 2}} \|D^\alpha \omega D^\beta (u-\eta)\|_{L_p(V^+)}\Big). \end{aligned} \tag{2.207}$$

Since η is a polynomial of degree at most $k-2$, we see by Equations 2.200 and 2.204 that

$$\begin{aligned} \|\omega D^\zeta B_1 \eta\|_{W_p^1(V^+)} &\le \|\omega\|_{W_\infty^1(V^+)} \|\eta\|_{W_p^{k-1}(V^+)} \\ &\le C d^{-1}\Big(\|u-\eta\|_{W_p^{k-1}(V^+)} + \|u\|_{W_p^{k-1}(V^+)}\Big) \\ &\le C d^{-1} \|u\|_{W_p^{k-1}(V^+)}. \end{aligned} \tag{2.208}$$

If $|\alpha|+|\beta|\le k$ and $|\alpha|\le 2$ then, by Equations 2.200 and 2.204,

$$\begin{aligned} \|D^\alpha \omega D^\beta (u-\eta)\|_{L_p(V^+)} &\le |\omega|_{W_\infty^{|\alpha|}(V^+)} |u-\eta|_{W_p^{|\beta|}(V^+)} \\ &\le C d^{-|\alpha|} d^{(k-1)-|\beta|} |u|_{W_p^{k-1}(V^+)} \\ &= C d^{-1} |u|_{W_p^{k-1}(V^+)}. \end{aligned} \tag{2.209}$$

Equations 2.207, 2.208, and 2.209 prove Equation 2.201. Now let $\chi = D^\zeta \eta$. By Equation 2.204, if $i \in 0:1$ then

$$\begin{aligned} |D^\zeta u - \chi|_{W_p^i(V^+)} &\le |u-\eta|_{W_p^{k-2+i}(V^+)} \\ &\le C d^{(k-1)-(k-2+i)} |u|_{W_p^{k-1}(V^+)} \\ &= C d^{1-i} |u|_{W_p^{k-1}(V^+)}, \end{aligned} \tag{2.210}$$

verifying Equation 2.202. Also observe that, on T,

$$B_1(\omega(D^\zeta u - \chi)) = B_1(\omega D^\zeta (u-\eta)) = v_1. \tag{2.211}$$

Now we rejoin the cases $m_1 = 0$ and $m_1 = 1$. Since $B_1(\omega(D^\zeta u - \chi)) = v_1$ on $\partial\Omega$, we have by Theorem 2.1 that

$$\begin{aligned} |D^\zeta u|_{W_p^2(U^+)} &= |\omega(D^\zeta u - \chi)|_{W_p^2(U^+)} \\ &\leq C\Big(\|L(\omega D^\zeta (u-\chi))\|_{L_p(V^+)} \\ &\qquad + \|v_1\|_{W_p^{2-m_1}(V^+)} \\ &\qquad + \|\omega D^\zeta (u-\chi)\|_{W_p^1(V^+)}\Big). \end{aligned} \tag{2.212}$$

Here we have used the fact that $C_p \leq 2N$. By the general Leibniz rule,

$$L(\omega(D^\zeta u - \chi)) = \omega L(D^\zeta u - \chi) + \sum_{\substack{|\alpha|+|\beta|\leq 2 \\ |\beta|<2}} c_{2,\alpha,\beta} D^\alpha \omega D^\beta (D^\zeta u - \chi), \tag{2.213}$$

where the $c_{2,\alpha,\beta} : \bar{\Omega} \to \mathbb{R}$ can be expressed in terms of ζ and the coefficients of L. Since $Lu = 0$ on V^+, we have by Proposition 2.6 that

$$\|LD^\zeta u\|_{L_p(V^+)} \leq C\|u\|_{W_p^{k-1}(V^+)}. \tag{2.214}$$

Using the fact that χ is a polynomial of degree 0, along with Equation 2.202,

$$\begin{aligned} \|L\chi\|_{L_p(V^+)} &\leq C\|\chi\|_{W_p^2(V^+)} \\ &= C\|\chi\|_{W_p^1(V^+)} \\ &\leq C\Big(\|D^\zeta u - \chi\|_{W_p^1(V^+)} + \|D^\zeta u\|_{W_p^1(V^+)}\Big) \\ &\leq C\|u\|_{W_p^{k-1}(V^+)}. \end{aligned} \tag{2.215}$$

If $|\alpha| + |\beta| \leq 2$ and $|\beta| < 2$ then, by Equations 2.200 and 2.202

$$\begin{aligned} \|D^\alpha \omega D^\beta (D^\zeta u - \chi)\|_{L_p(V^+)} &\leq |\omega|_{W_\infty^{|\alpha|}(V^+)} |D^\zeta u - \chi|_{W_p^{|\beta|}(V^+)} \\ &\leq C d^{-|\alpha|} d^{1-|\beta|} |u|_{W_p^{k-1}(V^+)} \\ &= C d^{-1} |u|_{W_p^{k-1}(V^+)}. \end{aligned} \tag{2.216}$$

By Equations 2.213, 2.214, 2.215, and 2.216,

$$\|L(\omega(D^\zeta u - \chi))\|_{L_p(V^+)} \leq Cd^{-1}\|u\|_{W_p^{k-1}(V^+)}. \tag{2.217}$$

By Equation 2.216,

$$\begin{aligned} \|\omega(D^\zeta u - \chi)\|_{W_p^1(V^+)} &\leq C \sum_{|\alpha|+|\beta|\leq 1} \|D^\alpha \omega D^\beta (D^\zeta u - \chi)\|_{L_p(V^+)} \\ &\leq Cd^{-1}\|u\|_{W_p^{k-1}(V^+)}. \end{aligned} \tag{2.218}$$

By Equations 2.212, 2.217, 2.201, and 2.218,

$$|D^\zeta u|_{W_p^2(U^+)} \leq Cd^{-1}\|u\|_{W_p^{k-1}(V^+)}. \tag{2.219}$$

The lemma follows by Proposition 2.7. $\square$

Lemma 2.19. *Suppose that $x_0 \in \partial\Omega$ and $d > 0$. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{2d}(x_0) \cap \Omega$, and $T = B_{2d}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$. If $k \geq 2$, $u \in W_1^k(V^+)$, $Lu = 0$ on V^+, and $B_1 u = 0$ on T then*

$$|u|_{W^k_{\frac{N}{N-1}}(U^+)} \leq Cd^{-1}\|u\|_{W_1^k(V^+)}. \tag{2.220}$$

Proof. Define $U = B_d(x_0)$ and $V = B_{2d}(x_0)$.

Let $\omega \in C_0^\infty(V)$ be such that $\omega = 1$ on U and, for $i \in 0:k$,

$$|\omega|_{W^i_\infty(V)} \leq Cd^{-i}. \tag{2.221}$$

Let $|\zeta| = k-2$ have $\zeta_N = 0$. Our first goal is to show that there exist $\chi \in \Pi^1(V^+)$ and $v_1 \in W_p^{2-m_1}(\Omega)$ such that $B_1(\omega(D^\zeta u - \chi)) = v_1$ on $\partial\Omega$,

$$\|v_1\|_{W^{2-m_1}_{\frac{N}{N-1}}(V^+)} \leq Cd^{-1}\|u\|_{W_1^k(V^+)}, \tag{2.222}$$

and, for $i \in 0:2$,

$$|D^\zeta u - \chi|_{W_1^i(V^+)} \leq Cd^{2-i}|u|_{W_1^k(V^+)}. \tag{2.223}$$

First we consider the case $m_1 = 0$. By definition of m_1, if $v : \bar{\Omega} \to \mathbb{R}$ then $B_1 v = b_0 v$ for some $b_0 : \bar{\Omega} \to \mathbb{R}$. By the complementing condition, we must have $b_0(x) \neq 0$ for all $x \in \partial\Omega$. Therefore, $u = 0$ on $\partial\Omega$. Let v be the extension of $D^\zeta u$ from V^+ to V that is odd in its Nth argument. On T, $D^\zeta u$ is a tangential derivative of a function which is zero on T, and is thus itself zero on T. Therefore, $v \in W_1^2(V)$. If $i \in 1 : N-1$ then $D_i v$ is odd in its Nth argument. Let c be the average value of $D_N v$ on V. Here we take $\chi(x) = c x_N$ and $v_1 = 0$. Notice that χ is odd in its Nth argument, $D_i \chi = 0$ for $i \in 1 : N-1$, and $D_N \chi = c$. It is now clear that, for $|\alpha| \leq 1$, $D^\alpha(v - \chi)$ has average value 0 on V. Equation 2.223 follows by Poincaré's inequality. Also observe that, on T, $\chi = 0$, so

$$B_1(\omega(D^\zeta u - \chi)) = b_0 \omega(D^\zeta u - \chi) = 0 = v_1. \tag{2.224}$$

Equation 2.222 is obvious.

Next we consider the case $m_1 = 1$. By the Bramble-Hilbert lemma, there exists some $\eta \in \Pi^{k-1}(V^+)$ such that, if $i \in 0 : k$ then

$$|u - \eta|_{W_1^i(V^+)} \leq C d^{k-i} |u|_{W_1^k(V^+)}. \tag{2.225}$$

By the general Leibniz rule,

$$B_1(\omega D^\zeta(u - \eta)) = \omega D^\zeta B_1(u - \eta) + \sum_{\substack{|\alpha|+|\beta| \leq k-1 \\ |\alpha| \leq 1}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (u - \eta), \tag{2.226}$$

where the $c_{1,\alpha,\beta} : \bar{\Omega} \to \mathbb{R}$ can be expressed in terms of ζ and the coefficients of B_1. On T, $D^\zeta B_1 u$ is a tangential derivative of a function which is zero on T, and is thus itself zero on T. Therefore, if we define

$$v_1 = -\omega D^\zeta B_1 \eta + \sum_{\substack{|\alpha|+|\beta| \leq k-1 \\ |\alpha| \leq 1}} c_{1,\alpha,\beta} D^\alpha \omega D^\beta (u - \eta), \tag{2.227}$$

then, by Equation 2.226, $B_1(\omega D^\zeta(u-\eta)) = v_1$ on $\partial\Omega$. Observe that

$$\begin{aligned}\|v_1\|_{W^1_{\frac{N}{N-1}}(V^+)} \leq C\Big(&\|\omega D^\zeta B_1\eta\|_{W^1_{\frac{N}{N-1}}(V^+)} \\ &+ \sum_{\substack{|\alpha|+|\beta|\leq k \\ |\alpha|\leq 2}} \|D^\alpha \omega D^\beta (u-\eta)\|_{L_{\frac{N}{N-1}}(V^+)}\Big).\end{aligned} \tag{2.228}$$

By Equations 2.221 and a scaled Sobolev inequality,

$$\begin{aligned}\|\omega D^\zeta B_1\eta\|_{W^1_{\frac{N}{N-1}}(V^+)} &\leq \sum_{i=0}^{1} \|\omega\|_{W^i_\infty(V^+)} |D^\zeta B_1\eta|_{W^{1-i}_{\frac{N}{N-1}}(V^+)} \\ &\leq C\sum_{i=0}^{1} d^{-i}\Big(d^{-1}|D^\zeta B_1\eta|_{W_1^{1-i}(V^+)} \\ &\quad + |D^\zeta B_1\eta|_{W_1^{2-i}(V^+)}\Big) \\ &\leq C\sum_{i=0}^{2} d^{-i}|D^\zeta B_1\eta|_{W_1^{2-i}(V^+)}.\end{aligned} \tag{2.229}$$

For $i \in 0:1$, we can use the fact that η is a polynomial of degree at most $k-1$, along with Equation 2.225, to find that

$$\begin{aligned}|D^\zeta B_1\eta|_{W_1^{2-i}(V^+)} &\leq C\|\eta\|_{W_1^k(V^+)} \\ &\leq C\Big(\|u-\eta\|_{W_1^k(V^+)} + \|u\|_{W_1^k(V^+)}\Big) \\ &\leq C\|u\|_{W_1^k(V^+)}.\end{aligned} \tag{2.230}$$

For $i = 2$, this type of estimate will not suffice. Instead, we first observe that, by Equation 2.225,

$$\begin{aligned}\|D^\zeta B_1(u-\eta)\|_{L_1(V^+)} &\leq C\|u-\eta\|_{W_1^{k-1}(V^+)} \\ &\leq Cd|u|_{W_1^k(V^+)}.\end{aligned} \tag{2.231}$$

Since $D^\zeta B_1 u = 0$ on T, we see, using the mean value theorem, that

$$\begin{aligned}\|D^\zeta B_1 u\|_{L_1(V^+)} &\leq Cd|D^\zeta B_1 u|_{W_1^1(V^+)} \\ &\leq Cd\|u\|_{W_1^k(V^+)}.\end{aligned} \tag{2.232}$$

By Equations 2.231 and 2.232,

$$\begin{aligned}\|D^\zeta B_1\eta\|_{L_1(V^+)} &\leq \|D^\zeta B_1(u-\eta)\|_{L_1(V^+)} + \|D^\zeta B_1 u\|_{L_1(V^+)} \\ &\leq Cd\|u\|_{W_1^k(V^+)}.\end{aligned} \tag{2.233}$$

Putting together Equations 2.229, 2.230, and 2.233, we have that

$$\|\omega D^\zeta B_1\eta\|_{W^1_{\frac{N}{N-1}}(V^+)} \leq Cd^{-1}\|u\|_{W_1^k(V^+)}. \tag{2.234}$$

If $|\alpha| + |\beta| \leq k$ and $|\alpha| \leq 2$ then, by a scaled Sobolev inequality and Equations 2.221 and 2.225,

$$\begin{aligned}\|D^\alpha\omega D^\beta(u-\eta)\|_{L_{\frac{N}{N-1}}(V^+)} &\leq |\omega|_{W_\infty^{|\alpha|}(V^+)}|u-\eta|_{W^{|\beta|}_{\frac{N}{N-1}}(V^+)} \\ &\leq Cd^{-|\alpha|}\Big(d^{-1}|u-\eta|_{W_1^{|\beta|}(V^+)} \\ &\qquad + |u-\eta|_{W_1^{|\beta|+1}(V^+)}\Big) \\ &\leq Cd^{-|\alpha|}\Big(d^{-1}d^{k-|\beta|}|u|_{W_1^k(V^+)} \\ &\qquad + d^{k-(|\beta|+1)}|u|_{W_1^k(V^+)}\Big) \\ &\leq Cd^{-1}|u|_{W_1^k(V^+)}.\end{aligned} \tag{2.235}$$

Equations 2.228, 2.234, and 2.235 prove Equation 2.222. Now let $\chi = D^\zeta\eta$. By Equation 2.225, if $i \in 0:2$ then

$$\begin{aligned}|D^\zeta u - \chi|_{W_1^i(V^+)} &\leq |u-\eta|_{W_1^{k-2+i}(V^+)} \\ &\leq Cd^{k-(k-2+i)}|u|_{W_1^k(V^+)} \\ &= Cd^{2-i}|u|_{W_1^k(V^+)},\end{aligned} \tag{2.236}$$

verifying Equation 2.223. Also observe that, on T,

$$B_1(\omega(D^\zeta u - \chi)) = B_1(\omega D^\zeta(u-\eta)) = v_1. \tag{2.237}$$

Now we rejoin the cases $m_1 = 0$ and $m_1 = 1$. Since $B_1(\omega(D^\zeta u - \chi)) = v_1$ on

$\partial\Omega$, we have by Theorem 2.1 that

$$\begin{aligned} |D^\zeta u|_{W^2_{\frac{N}{N-1}}(U^+)} &= |\omega(D^\zeta u - \chi)|_{W^2_{\frac{N}{N-1}}(U^+)} \\ &\le C\Big(\|L(\omega D^\zeta(u-\chi))\|_{L_{\frac{N}{N-1}}(V^+)} \\ &\qquad + \|v_1\|_{W^{2-m_1}_{\frac{N}{N-1}}(V^+)} \\ &\qquad + \|\omega D^\zeta(u-\chi)\|_{W^1_{\frac{N}{N-1}}(V^+)}\Big). \end{aligned} \tag{2.238}$$

By the general Leibniz rule,

$$L(\omega(D^\zeta u - \chi)) = \omega L(D^\zeta u - \chi) + \sum_{\substack{|\alpha|+|\beta|\le 2 \\ |\beta|<2}} c_{2,\alpha,\beta} D^\alpha \omega D^\beta (D^\zeta u - \chi), \tag{2.239}$$

where the $c_{2,\alpha,\beta} : \bar{\Omega} \to \mathbb{R}$ can be expressed in terms of ζ and the coefficients of L. Since $Lu = 0$ on V^+ we have, by Proposition 2.6 and a scaled Sobolev inequality, that

$$\begin{aligned} \|LD^\zeta u\|_{L_{\frac{N}{N-1}}(V^+)} &\le C\|u\|_{W^{k-1}_{\frac{N}{N-1}}(V^+)} \\ &\le Cd^{-1}\|u\|_{W^k_1(V^+)}. \end{aligned} \tag{2.240}$$

Using the fact that χ is a polynomial of degree at most 1, along with a scaled Sobolev inequality and Equation 2.225,

$$\begin{aligned} \|L\chi\|_{L_{\frac{N}{N-1}}(V^+)} &\le C\|\chi\|_{W^1_{\frac{N}{N-1}}(V^+)} \\ &= Cd^{-1}\|\chi\|_{W^2_1(V^+)} \\ &\le Cd^{-1}\Big(\|D^\zeta u - \chi\|_{W^2_1(V^+)} + \|D^\zeta u\|_{W^2_1(V^+)}\Big) \\ &\le Cd^{-1}\|u\|_{W^k_1(V^+)}. \end{aligned} \tag{2.241}$$

If $|\alpha| + |\beta| \le 2$ and $|\beta| < 2$ then, by Equations 2.221 and 2.223 and a scaled

Sobolev inequality,

$$\begin{aligned}\|D^\alpha\omega D^\beta(D^\zeta u-\chi)\|_{L_{\frac{N}{N-1}}(V^+)} &\le |\omega|_{W_\infty^{|\alpha|}(V^+)}|D^\zeta u-\chi|_{W^{|\beta|}_{\frac{N}{N-1}}(V^+)}\\ &\le Cd^{-|\alpha|}\Big(d^{-1}|D^\zeta u-\chi|_{W_1^{|\beta|}(V^+)}\\ &\qquad+|D^\zeta u-\chi|_{W_1^{|\beta|+1}(V^+)}\Big)\\ &\le Cd^{-|\alpha|}\Big(d^{-1}d^{2-|\beta|}|u|_{W_1^k(V^+)}\\ &\qquad+d^{2-(|\beta|+1)}|u|_{W_1^k(V^+)}\Big)\\ &\le Cd^{-1}|u|_{W_1^k(V^+)}.\end{aligned} \tag{2.242}$$

By Equations 2.239, 2.240, 2.241, and 2.242,

$$\|L(\omega(D^\zeta u-\chi))\|_{L_{\frac{N}{N-1}}(V^+)} \le Cd^{-1}\|u\|_{W_1^k(V^+)}. \tag{2.243}$$

By Equation 2.242,

$$\begin{aligned}\|\omega(D^\zeta u-\chi)\|_{W^1_{\frac{N}{N-1}}(V^+)} &\le C\sum_{|\alpha|+|\beta|\le 1}\|D^\alpha\omega D^\beta(D^\zeta u-\chi)\|_{L_{\frac{N}{N-1}}(V^+)}\\ &\le Cd^{-1}\|u\|_{W_1^k(V^+)}.\end{aligned} \tag{2.244}$$

By Equations 2.238, 2.243, 2.222, and 2.244,

$$|D^\zeta u|_{W^2_{\frac{N}{N-1}}(U^+)} \le Cd^{-1}\|u\|_{W_1^k(V^+)}. \tag{2.245}$$

The lemma follows by Proposition 2.7. □

Lemma 2.20. *Suppose that $x_0\in\partial\Omega$ and $d>0$. Let $U^+=B_d(x_0)\cap\Omega$, $V^+=B_{2d}(x_0)\cap\Omega$, and $T=B_{2d}(x_0)\cap\partial\Omega$. Assume that $V^+\subset\mathbb{R}^N_+$ and $T\subset\partial\mathbb{R}^N_+$. If $k\ge 2$, $u\in W^k_{2N}(V^+)$, $Lu=0$ on V^+, and $B_1u=0$ on T then*

$$|u|_{W_\infty^{k-1}(U^+)} \le Cd^{-1/2}\|u\|_{W_{2N}^{k-1}(V^+)}. \tag{2.246}$$

Proof. By a scaled Sobolev inequality,

$$|u|_{W_\infty^{k-1}(U^+)} \le Cd^{1/2}\Big(d^{-1}\|u\|_{W_{2N}^{k-1}(U^+)}+|u|_{W_{2N}^k(U^+)}\Big). \tag{2.247}$$

By Lemma 2.19,

$$|u|_{W^k_{2N}(U^+)} \leq Cd^{-1}\|u\|_{W^{k-1}_{2N}(V^+)}. \tag{2.248}$$

Equations 2.247 and 2.248 prove the lemma. □

Lemma 2.21. *Suppose that $x_0 \in \partial\Omega$, $d > 0$, and $k \geq 2$. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{2^k d}(x_0) \cap \Omega$, and $T = B_{2^k d}(x_0) \cap \partial\Omega$. Assume that $V^+ \subset \mathbb{R}^N_+$ and $T \subset \partial\mathbb{R}^N_+$.*

1. *If $\frac{N}{N-1} \leq p \leq 2N$, $u \in W^k_p(V^+)$, $Lu = 0$ on V^+, $B_1 u = 0$ on T, and $\ell \in 1 : k$ then*

$$\|u\|_{W^k_p(U^+)} \leq Cd^{-(k-\ell)}\|u\|_{W^\ell_p(V^+)}. \tag{2.249}$$

2. *If $1 \leq p \leq \frac{N}{N-1}$, $u \in W^k_p(V^+)$, $Lu = 0$ on V^+, $B_1 u = 0$ on T, and $\ell \in 2 : k$ then*

$$\|u\|_{W^k_p(U^+)} \leq Cd^{-(k-\ell)}\|u\|_{W^\ell_p(V^+)}. \tag{2.250}$$

3. *If $2N \leq p \leq \infty$, $u \in W^k_{2N}(V^+)$, $Lu = 0$ on V^+, $B_1 u = 0$ on T, and $\ell \in 2 : k$ then*

$$\|u\|_{W^{k-1}_p(U^+)} \leq Cd^{-(k-\ell)}\|u\|_{W^{\ell-1}_p(V^+)}. \tag{2.251}$$

Proof. These results follow from Lemmas 2.18, 2.19, and 2.20 in the same way that Lemma 2.17 follows from Lemmas 2.14, 2.15, and 2.16. □

Lemma 2.22. *Suppose that $x_0 \in \partial\Omega$, $0 < d \leq d'$, and $k \geq 2$. Assume that d' and d/d' are sufficiently small. Let $U^+ = B_d(x_0) \cap \Omega$, $V^+ = B_{d'}(x_0) \cap \Omega$, and $T = B_{d'}(x_0) \cap \partial\Omega$.*

1. *If $\frac{N}{N-1} \leq p \leq 2N$, $u \in W^k_p(V^+)$, $Lu = 0$ on V^+, $B_1 u = 0$ on T, and $\ell \in 1 : k$ then*

$$\|u\|_{W^k_p(U^+)} \leq Cd^{-(k-\ell)}\|u\|_{W^\ell_p(V^+)}. \tag{2.252}$$

2. *If* $1 \leq p \leq \frac{N}{N-1}$, $u \in W_p^k(V^+)$, $Lu = 0$ *on* V^+, $B_1 u = 0$ *on* T, *and* $\ell \in 2:k$ *then*

$$\|u\|_{W_p^k(U^+)} \leq C d^{-(k-\ell)} \|u\|_{W_p^\ell(V^+)}. \tag{2.253}$$

3. *If* $2N \leq p \leq \infty$, $u \in W_{2N}^k(V^+)$, $Lu = 0$ *on* V^+, $B_1 u = 0$ *on* T, *and* $\ell \in 2:k$ *then*

$$\|u\|_{W_p^{k-1}(U^+)} \leq C d^{-(k-\ell)} \|u\|_{W_p^{\ell-1}(V^+)}. \tag{2.254}$$

Proof. Let $U = B_d(x_0)$ and $V = B_{d'}(x_0)$.

For sufficiently small d', there exists an invertible and sufficiently smooth $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ which flattens the boundary of Ω in V and has sufficiently smooth inverse. With $\hat{x}_0 = \Phi(x_0)$, $\hat{U}^+ = \Phi(U^+)$, $\hat{V}^+ = \Phi(V^+)$, $\hat{T} = \Phi(T)$, $\hat{U} = \Phi(U)$, and $\hat{V} = \Phi(V)$, this means that $\hat{V}^+ \subset \mathbb{R}_+^N$ and $\hat{T} \subset \partial\mathbb{R}_+^N$. If d/d' is sufficiently small, there exists some $\hat{d} > 0$ such that $\hat{U} \subset B_{\hat{d}}(\hat{x}_0)$ and $B_{2^k\hat{d}}(\hat{x}_0) \subset \hat{V}$. The results follow by applying Lemma 2.21 to the transformed setup. □

2.9.3 General Domains

In this subsection, we prove Theorem 2.4.

Let $d_{\text{int}} > 0$ be such that, if $d'_{\text{int}} = 2^k d_{\text{int}}$, $d_{\text{bdry}} = d'_{\text{int}}$, and $d'_{\text{bdry}} = d$, then $d'_{\text{bdry}} \geq d_{\text{bdry}}$. Define $\Omega_{\text{int}} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq d'_{\text{int}}\}$. For $x \in U \cap \Omega_{\text{int}}$, let $U_x = B_{d_{\text{int}}}(x)$ and $V_x = B_{d'_{\text{int}}}(x)$, and notice that $V_x \subset V \cap \Omega$. For $x \in U \cap \partial\Omega$, let $U_x = B_{d_{\text{bdry}}}(x) \cap \Omega$ and $V_x = B_{d'_{\text{bdry}}}(x) \cap \Omega$, and notice that $V_x \subset V \cap \Omega$.

There exist finite subsets X_1 and X_2 of $U \cap \Omega_{\text{int}}$ and $U \cap \partial\Omega$, respectively, such that $U \cap \Omega$ is covered by the open sets U_x for $x \in X_1 \cup X_2$ and no point of $V \cap \Omega$ is in more than C of the sets V_x for $x \in X_1 \cup X_2$. The sizes of the sets X_1 and X_2 are irrelevant.

We apply Lemma 2.17 with domains U_x and V_x for all $x \in X_1$. Taking

$d_{\mathrm{bdry}}/d'_{\mathrm{bdry}}$ sufficiently small, we apply Lemma 2.22 with domains U_x and V_x for all $x \in X_2$. Raising the results to the pth power, summing over all $x \in X_1 \cup X_2$, and taking pth roots, we obtain the theorem.

2.10 Appendix: Singular Integral Operators

Central to any proof of the L_p-based estimates for solutions of partial differential equations that relies on potential theory, including those of [2, Theorem 15.2], [12, Theorem 9.13], [29, Equation 2], and the one given here, are estimates for various singular integral operators. We make explicit the dependence on p in these estimates.

Theorem 2.23. *Suppose that* $K \in C^1(\mathbb{R}^N \setminus \{0\})$*,* $c > 0$*,* $|K(x)| \leq c|x|^{-N}$ *and* $|D_i K(x)| \leq c|x|^{-(N+1)}$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$ *and* $i \in 1 : N$*, and* $\int_{\Sigma^{N-1}} K(tx)\, dS(x) = 0$ *for all* $t > 0$*. If* $1 < p < \infty$ *and* $f \in L_p(\mathbb{R}^N)$ *then*

$$\|K \hat{*} f\|_{L_p(\mathbb{R}^N)} \leq C C_p \|f\|_{L_p(\mathbb{R}^N)}, \tag{2.255}$$

where C *depends on* N *and* c*.*

Proof. This result is given by [27, Chapter II, Theorem 2]. The precise dependence on p is given by [27, Chapter II, Further Result 6.2(a)]. □

Theorem 2.24. *Suppose that* $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ *is odd and homogeneous of degree* $-N$*,* $c > 0$*, and* $\|K\|_{L_1(\Sigma^{N-1})} \leq c$*. If* $1 < p < \infty$ *and* $f \in L_p(\mathbb{R}^N)$ *then*

$$\|K \hat{*} f\|_{L_p(\mathbb{R}^N)} \leq C C_p \|f\|_{L_p(\mathbb{R}^N)}, \tag{2.256}$$

where C *depends on* N *and* c*.*

Proof. This result is given by [6, Theorem 3]. The precise dependence on p can be traced through the proof of this theorem. It is easily seen to be the same as that of the norm of the Hilbert transform on $L_p(\mathbb{R}^N)$, which we know from Theorem 2.23. $\square$

In the results presented thus far, we have found estimates on all of $\mathbb{R}^N$. However, we are also interested in estimates on bounded domains. Furthermore, some of the functions we are operating on are supported in a bounded domain away from the domain of interest. The following corollary and the results which follow from it take these possibilities into account and give sharper estimates than we would have otherwise.

Corollary 2.25. *Assume that* $0 < d_1 \leq d_2$ *and* V, W *are open subsets of* $\mathbb{R}^N$ *such that, if* $x \in W$ *and* $y \in V$, *then* $d_1 \leq |x-y| \leq d_2$. *Let* $1 < p < \infty$, $g, h \in L_p(\mathbb{R}^N)$, *assume that* $h = 0$ *outside of* V, *and let* $f = g + h$.

1. *Suppose that* $G \in C^2(\mathbb{R}^N \setminus \{0\})$ *is homogeneous of degree* $-(N-1)$, $c > 0$, *and* $\|G\|_{W^2_\infty(\Sigma^{N-1})} \leq c$. *If* $i \in 1 : N$ *and* $K = D_i G$ *then*

$$\|K \hat{*} f\|_{L_p(W)} \leq C\Big(C_p \|g\|_{L_p(\mathbb{R}^N)} + \|h\|_{L_p(V)}\Big), \tag{2.257}$$

   where C *depends on* N, c, *and* d_2/d_1.

2. *If* $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ *is odd and homogeneous of degree* $-N$, $c > 0$, *and* $\|K\|_{L_1(\Sigma^{N-1})} \leq c$ *then*

$$\|K \hat{*} f\|_{L_p(W)} \leq C\Big(C_p \|g\|_{L_p(\mathbb{R}^N)} + \|h\|_{L_p(V)}\Big), \tag{2.258}$$

   where C *depends on* N, c, *and* d_2/d_1.

Proof. Define $U = \{x - y : x \in W, y \in V\}$. By assumption, $U \subset B_{d_2}(0) \setminus B_{d_1}(0)$. By Young's inequality, $\|K * h\|_{L_p(W)} \leq \|K\|_{L_1(U)} \|h\|_{L_p(V)}$. Therefore, to prove the corollary, it remains only to estimate $\|K\|_{L_1(U)}$ and $\|K \hat{*} g\|_{L_p(W)}$.

First consider part 1. By [3, p. 223], $\int_{\Sigma^{N-1}} K(tx)\, dS(x) = 0$ for all $t > 0$. Obviously $K \in C^1(\mathbb{R}^N \setminus \{0\})$ is homogeneous of degree $-N$ and the first derivatives of K are homogeneous of degree $-(N+1)$. Furthermore, $|K(x)| \leq c|x|^{-N}$ and $|D_j K(x)| \leq c|x|^{-(N+1)}$ for all $x \in \mathbb{R}^N \smallsetminus \{0\}$ and $j \in 1 : N$. Therefore,

$$\begin{aligned}
\|K\|_{L_1(U)} &\leq c \int_{d_1<|x|<d_2} |x|^{-N}\, dx \\
&\leq C \int_{d_1}^{d_2} t^{-N} t^{N-1}\, dt \\
&= C \log(d_2/d_1),
\end{aligned} \tag{2.259}$$

where C depends on N and c. This, along with Theorem 2.23, gives part 1.

In the case of part 2, we see that

$$\begin{aligned}
\|K\|_{L_1(U)} &\leq \int_{d_1<|x|<d_2} |K(x)|\, dx \\
&= \int_{d_1}^{d_2} \int_{\Sigma^{N-1}} |K(tx)| t^{N-1}\, dS(x)\, dt \\
&= \Big(\int_{d_1}^{d_2} t^{-N} t^{N-1}\, dt \Big) \Big(\int_{\Sigma^{N-1}} |K(x)|\, dS(x) \Big) \\
&\leq c \log(d_2/d_1).
\end{aligned} \tag{2.260}$$

This, along with Theorem 2.24, gives part 2. $\square$

In the remainder of this appendix, we revise various estimates of [2, Section 3]. First we have the analogue of [2, Lemma 3.2].

Lemma 2.26. *Suppose that $K \in C^0(\mathbb{R}^N_+)$ is homogeneous of degree $-N$, $c > 0$, and $\|K\|_{L_1(\Sigma^{N-1}_+)} \leq c$. Assume that $0 < d_1 \leq d_2$ and V, W are open subsets of $\mathbb{R}^N_+$ such that, if $x \in W$ and $y \in V$, then $d_1 \leq |x - y^*| \leq d_2$. Let $1 < p < \infty$, $g, h \in L_p(\mathbb{R}^N_+)$, assume that $h = 0$ outside of V, and let $f = g + h$. For $x \in \mathbb{R}^{N-1}$ and $t > 0$, define*

$$u(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} K(x-y, t+s) f(y,s)\, ds\, dy. \tag{2.261}$$

Then

$$\|u\|_{L_p(W)} \leq C \Big(C_p \|g\|_{L_p(\mathbb{R}^N_+)} + \|h\|_{L_p(V)} \Big), \tag{2.262}$$

where C depends on N, c and d_2/d_1.

Proof. Let $\bar{K}$ be the odd extension of $|K|$ from $\mathbb{R}^N_+$ to $\mathbb{R}^N$. This differs from the treatment in the proof of [2, Lemma 3.2], where $\bar{K}$ is taken to be the extension that is odd in its Nth argument. Notice that $\bar{K}$ is homogeneous of degree $-N$ and $\|\bar{K}\|_{L_1(\Sigma^{N-1})} \le 2c$. For $x \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$, define

$$\bar{g}(x,t) = \begin{cases} 0, & \text{if } t \ge 0 \\ |g(x,-t)|, & \text{if } t < 0 \end{cases} \tag{2.263}$$

and

$$\bar{h}(x,t) = \begin{cases} 0, & \text{if } t \ge 0 \\ |h(x,-t)|, & \text{if } t < 0. \end{cases} \tag{2.264}$$

We point out that $\bar{h} = 0$ outside of V^*. Let $\bar{f} = \bar{g} + \bar{h}$ and $\bar{u} = \bar{K} \hat{*} \bar{f}$. Then, for $x \in \mathbb{R}^{N-1}$ and $t > 0$,

$$\begin{aligned} |u(x,t)| &\le \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}} \bar{K}(x-y, t+s) \bar{f}(y,-s)\, ds\, dy \\ &= \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}} \bar{K}(x-y, t-s) \bar{f}(y,s)\, ds\, dy \\ &= \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{|t-s|>\epsilon} \bar{K}(x-y, t-s) \bar{f}(y,s)\, ds\, dy \\ &= \bar{u}(x,t). \end{aligned} \tag{2.265}$$

By Corollary 2.25, Part 2,

$$\begin{aligned} \|u\|_{L_p(W)} &\le \|\bar{u}\|_{L_p(W)} \\ &\le C\Big(C_p \|\bar{g}\|_{L_p(\mathbb{R}^N)} + \|\bar{h}\|_{L_p(V^*)}\Big), \end{aligned} \tag{2.266}$$

where C depends on N, c and d_2/d_1. Since $\|\bar{g}\|_{L_p(\mathbb{R}^N)} = \|g\|_{L_p(\mathbb{R}^N_+)}$ and $\|\bar{h}\|_{L_p(V^*)} = \|h\|_{L_p(V)}$, this proves the lemma. $\square$

Next we have the analogue of [2, Theorem A3.1].

Lemma 2.27. *Suppose that $K \in C^1(\mathbb{R}^N_+)$ is homogeneous of degree $-(N-1)$, $c > 0$, and $\|K\|_{W^1_1(\Sigma^{N-1}_+)} \le c$. Assume that $0 < d_1 \le d_2$ and V, W are open subsets* *of $\mathbb{R}^N_+$ such that, if $x \in W$ and $y \in V$, then $d_1 \le |x - y^*| \le d_2$. Let $1 < p < \infty$ and $f \in W^1_p(\mathbb{R}^N_+)$. For $i \in 1 : N$, let $g_i, h_i \in W^1_p(\mathbb{R}^N_+)$ be such that $h_i = 0$ outside of V and $D_i f = g_i + h_i$. For $x \in \mathbb{R}^{N-1}$ and $t > 0$, define*

$$u(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} K(x-y,t) f(y,0)\, dy. \tag{2.267}$$

If $i \in 1 : N-1$ then

$$\|D_i u\|_{L_p(W)} \le C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}^N_+)} + \|h_j\|_{L_p(V)} \Big), \tag{2.268}$$

where C depends on N, c and d_2/d_1.

Proof. It is shown in the proof of [2, Theorem A3.1] that, for $x \in \mathbb{R}^{N-1}$ and $t > 0$,

$$\begin{aligned} -D_i u(x,t) = {} & \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} D_i K(x-y, t+s) D_N f(y,s)\, ds\, dy \\ & + \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} D_N K(x-y, t+s) D_i f(y,s)\, ds\, dy. \end{aligned} \tag{2.269}$$

For $j \in 1 : N$, $D_j K$ is homogeneous of degree $-N$. Therefore, by Lemma 2.26,

$$\begin{aligned} \|D_i u\|_{L_p(W)} \le C\Big(& C_p \big(\|g_N\|_{L_p(\mathbb{R}^N_+)} + \|g_i\|_{L_p(\mathbb{R}^N_+)} \big) \\ & + \|h_N\|_{L_p(V)} + \|h_i\|_{L_p(V)} \Big), \end{aligned} \tag{2.270}$$

where C depends on N, c, and d_2/d_1. □

Next we have the analogue of [2, Lemma A3.1].

Lemma 2.28. *Suppose that $K \in C^0(\mathbb{R}^N_+)$ is homogeneous of degree $-(N+1)$, $c > 0$, $\|K\|_{L_\infty(\Sigma^{N-1}_+)} \le c$, and*

$$\int_{\mathbb{R}^N} K(x,t)\, dx = 0 \tag{2.271}$$

for all $t > 0$. Assume that $0 < d_1 \le d_2$ and V, W are open subsets of $\mathbb{R}^N_+$ such that, if $x \in W$ and $y \in V$, then $d_1 \le |x - y^*| \le d_2$. Let $1 < p < \infty$ and

$f \in W_p^1(\mathbb{R}_+^N)$. For $i \in 1:N$, let $g_i, h_i \in W_p^1(\mathbb{R}_+^N)$ be such that $h_i = 0$ outside of V and $D_i f = g_i + h_i$. For $x \in \mathbb{R}^{N-1}$ and $t > 0$, define

$$u(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} K(x-y, t+s) f(y,s)\, dy. \tag{2.272}$$

Then

$$\|u\|_{L_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}_+^N)} + \|h_j\|_{L_p(V)} \Big), \tag{2.273}$$

where C depends on N, c and d_2/d_1.

Proof. First consider the case $N = 2$. It is shown in the proof of [2, Lemma A3.1] that, if $x \in \mathbb{R}$ and $t > 0$, then

$$u(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} \bar{K}(x-y, t+s) D_1 f(y,s)\, ds\, dy, \tag{2.274}$$

where $\bar{K} \in C^0(\mathbb{R}_+^2)$ is homogeneous of degree -2 and $\|\bar{K}\|_{L_1(\Sigma_+^1)}$ is bounded by a constant that depends on c. The lemma follows by applying Lemma 2.26.

From now on we assume that $N \geq 3$. It is shown in the proof of [2, Lemma A3.1] that

$$u = \sum_{i=1}^{N-1} (c_i v_i + w_i), \tag{2.275}$$

where the c_i are constants that can be bounded in terms of N and c, and, for $i \in 1 : N-1$, $x \in \mathbb{R}^{N-1}$, and $t > 0$,

$$v_i(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} \bar{K}(x-y, t+s) D_i f(y,s)\, ds\, dy \tag{2.276}$$

and

$$w_i(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} \bar{K}_i(x-y, t+s) D_i f(y,s)\, ds\, dy. \tag{2.277}$$

Here, $\bar{K} \in C^0(\mathbb{R}_+^N)$ is homogeneous of degree $-N$ and $\|\bar{K}\|_{L_1(\Sigma_+^{N-1})}$ is bounded by a constant that depends on N. Also, for $i \in 1 : N-1$, $\bar{K}_i \in C^0(\mathbb{R}_+^N)$ is homogeneous of degree $-N$ and, for $x \in \mathbb{R}^{N-1}$ and $t > 0$ with $(x,t) \in \Sigma_+^{N-1}$,

$$|\bar{K}_i(x,t)| \leq C\Big(1 + \log \frac{1}{t}\Big), \tag{2.278}$$

where C depends on N and c. If $i \in 1 : N-1$ then, by Lemma 2.26,

$$\|v_i\|_{L_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}^N_+)} + \|h_j\|_{L_p(V)} \Big), \tag{2.279}$$

where C depends on N, c, and d_2/d_1. If $i \in 1 : N-1$ then, since $t \mapsto \log \frac{1}{t}$ is integrable at 0, $\|\bar{K}_i\|_{L_1(\Sigma^{N-1}_+)}$ is bounded by a constant that depends on N and c. Therefore, by Lemma 2.26,

$$\|w_i\|_{L_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}^N_+)} + \|h_j\|_{L_p(V)} \Big), \tag{2.280}$$

where C depends on N, c, and d_2/d_1.

The lemma follows from Equations 2.275, 2.279, and 2.280 $\square$

Finally we have the analogue of [2, Theorem 3.3].

Theorem 2.29. *Suppose that $K \in C^2(\mathbb{R}^N_+)$ is homogeneous of degree $-(N-1)$, $c > 0$, $\|K\|_{W^2_\infty(\Sigma^{N-1}_+)} \leq c$, and*

$$\int_{\Sigma^{N-2}} K(x,0)\, dS(x) = 0. \tag{2.281}$$

Assume that $0 < d_1 \leq d_2$ and V, W are open subsets of $\mathbb{R}^N_+$ such that, if $x \in W$ and $y \in V$, then $d_1 \leq |x - y^*| \leq d_2$. Let $1 < p < \infty$ and $f \in W^1_p(\mathbb{R}^N_+)$. For $i \in 1 : N$, let $g_i, h_i \in W^1_p(\mathbb{R}^N_+)$ be such that $h_i = 0$ outside of V and $D_i f = g_i + h_i$. For $x \in \mathbb{R}^{N-1}$ and $t > 0$, define

$$u(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} K(x-y,t) f(y,0)\, dy. \tag{2.282}$$

Then

$$|u|_{W^1_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}^N_+)} + \|h_j\|_{L_p(V)} \Big), \tag{2.283}$$

where C depends on N, c and d_2/d_1.

Proof. Since $\|K\|_{W_1^1(\Sigma_+^{N-1})}$ is bounded by a constant that depends on N and c, if $i \in 1 : N-1$ then, by Lemma 2.27,

$$\|D_i u\|_{L_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}_+^N)} + \|h_j\|_{L_p(V)} \Big), \tag{2.284}$$

where C depends on N, c, and d_2/d_1. It is shown in the proof of [2, Theorem 3.3] that

$$-D_N u = v + w, \tag{2.285}$$

where, for $x \in \mathbb{R}^{N-1}$ and $t > 0$,

$$v(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} D_N K(x-y, t+s) D_N f(y,s)\, ds\, dy \tag{2.286}$$

and

$$w(x,t) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \int_{\mathbb{R}_+} D_N^2 K(x-y, t+s) f(y,s)\, ds\, dy. \tag{2.287}$$

Since $D_N K \in C^0(\mathbb{R}_+^N)$ is homogeneous of degree $-N$ and $\|D_N K\|_{L_1(\Sigma_+^{N-1})}$ is bounded by a constant that depends on N and c, we have by Lemma 2.26 that

$$\|v\|_{L_p(W)} \leq C \Big(C_p \|g_N\|_{L_p(\mathbb{R}_+^N)} + \|h_N\|_{L_p(V)} \Big), \tag{2.288}$$

where C depends on N, c, and d_2/d_1. It is shown in the proof of [2, Theorem 3.3] that $\int_{\mathbb{R}^N} D_N^2 K(y,t)\, dy = 0$ for all $t > 0$. Since, in addition, $D_N^2 K \in C^0(\mathbb{R}_+^N)$ is homogeneous of degree $-(N+1)$ and $\|D_N^2 K\|_{L_\infty(\Sigma_+^N)} \leq c$, we have by Lemma 2.28 that

$$\|w\|_{L_p(W)} \leq C \sum_{j=1}^{N} \Big(C_p \|g_j\|_{L_p(\mathbb{R}_+^N)} + \|h_j\|_{L_p(V)} \Big), \tag{2.289}$$

where C depends on N, c, and d_2/d_1. Putting together Equations 2.284, 2.285, 2.288, and 2.289 gives the result. $\square$

2.11 Future Work

The most egregious aspect of Theorem 2.1 is the factor of C_p^5 multiplying the $\|u\|_{W_p^{k-1}(\Omega)}$ on the right side of the estimate, and it is not clear how to avoid this.

Ideally, Theorems 2.2 and 2.3 could be made more general. However, these rely on the estimates of Section 2.6.2. At the crux of these arguments are two facts. First, any second-order differential operator can be linearly transformed into an operator whose leading part, at a certain point, is the Laplacian. Second, we know explicit representations of the solution of Poisson's equation in the half-space with homogeneous Dirichlet or constant coefficient oblique derivative boundary conditions. There is no obvious generalisation of these ideas to higher-order equations and multiple boundary conditions.

The local estimates on domains in the interior of Section 2.9.1 make no restrictions on the order of the differential operators, but the local estimates on domains at the boundary of Section 2.9.2 do. If such restrictions could be lifted, Theorem 2.4 could be made more general.

The estimates of Theorem 2.5 involve Sobolev and Lebesgue space norms with the same exponent on the left and right sides. Generalising this to allow for different exponents on the left and right sides seems straightforward, but a listing of all the possible cases would be messy.

CHAPTER 3

L_∞-BASED NEGATIVE NORM ERROR ESTIMATES FOR THE FINITE ELEMENT METHOD

3.1 Introduction and Statement of Results

Let $N \geq 2$ be an integer and let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. For $i, j \in 1:N$, let $a_{i,j}, b_i, c : \bar{\Omega} \to \mathbb{R}$ be sufficiently smooth. Define the bilinear form A on functions $v, w : \Omega \to \mathbb{R}$ by

$$A(v,w) = \int_\Omega \Bigg(\sum_{i,j=1}^N a_{i,j} D_i v D_j w + \sum_{i=1}^N b_i D_i v w + cvw \Bigg). \tag{3.1}$$

We assume that A is coercive over $W_2^1(\Omega)$. That is, there exists a constant $C_{\mathrm{co}} > 0$ such that, if $v \in W_2^1(\Omega)$ then

$$A(v,v) \geq C_{\mathrm{co}} \|v\|^2_{W_2^1(\Omega)}. \tag{3.2}$$

We also assume that A is uniformly elliptic on Ω. That is, there exists a constant $C_{\mathrm{ell}} > 0$ such that, if $x \in \Omega$ and $\xi \in \mathbb{R}^N$ then

$$\sum_{i,j=1}^N a_{i,j}(x) \xi_i \xi_j \geq C_{\mathrm{ell}} |\xi|^2. \tag{3.3}$$

Let $h > 0$ be sufficiently small, let $r \geq 2$ be an integer, and let $C_{\mathrm{ap}}, C_{\mathrm{sep}} > 0$. Let S_h be a finite-dimensional subspace of $W_\infty^1(\Omega)$ and let $I_h : W_\infty^1(\Omega) \to S_h$ be a projection. Assume that the following hold whenever U_1 and U_2 are open subsets of $\mathbb{R}^N$ with $U_1 \subset U_2$ and $\mathrm{dist}(U_1, \partial U_2) \geq C_{\mathrm{sep}} h$.

1. If $1 \leq p \leq \infty$, $k \in 0:1$, $\ell \in k:r$, and $v \in W_\infty^1(\Omega) \cap W_p^\ell(U_2 \cap \Omega)$ then

$$|v - I_h v|_{W_p^k(U_1 \cap \Omega)} \leq C_{\mathrm{ap}} h^{\ell - k} \|v\|_{W_p^\ell(U_2 \cap \Omega)}. \tag{3.4}$$

2. If $1 \le q \le p \le \infty$, $\ell \in -r : 0$, and $\chi \in S_h$ then

$$\|\chi\|_{L_p(U_1\cap\Omega)} \le C_{\mathrm{ap}} h^{-\ell - N(\frac{1}{q}-\frac{1}{p})} \|\chi\|_{W_q^{\ell}(U_2\cap\Omega)}, \tag{3.5}$$

and, for $i \in 1 : N$,

$$\|D_i\chi\|_{L_p(U_1\cap\Omega)} \le C_{\mathrm{ap}} h^{-\ell - N(\frac{1}{q}-\frac{1}{p})} \|D_i\chi\|_{W_q^{\ell}(U_2\cap\Omega)}. \tag{3.6}$$

3. If $k \in 0 : 1$, $\omega \in C^\infty(U_2)$, and $\chi \in S_h$ then

$$|\omega\chi - I_h(\omega\chi)|_{W_2^k(U_1\cap\Omega)} \le C_{\mathrm{ap}} h \sum_{i=0}^{k} |\omega|_{W_\infty^{k-i+1}(U_2)} |\chi|_{W_2^i(U_2\cap\Omega)}. \tag{3.7}$$

Let $u \in W_\infty^1(\Omega)$, $u_h \in S_h$, $F \in (W_2^1(\Omega))'$, and assume that

$$A(u - u_h, \chi) = F(\chi) \tag{3.8}$$

for all $\chi \in S_h$. Let $h \le H \le 1$ and let U be an open subset of $\mathbb{R}^N$ with $\mathrm{diam}(U) \le H$.

We will let C denote different positive constants that depend on N, Ω, various norms of the coefficients of A, C_{co}, C_{ell}, r, C_{ap}, and C_{sep}, in addition to other explicitly stated quantities.

The following theorem and its corollary are our main results.

Theorem 3.1. *If $k \in 1 : r-2$ and $r - 2 - k \le s \le r - 2$ then*

$$\begin{aligned} \|u - u_h\|_{W_\infty^{-k}(U,\Omega)} \le C\Big(& h^{k+1} \ell_{s=r-2,h,h/H} \Big(\inf_{\chi\in S_h} \|u - \chi\|_{W_\infty^1(\Omega),U,w,s} \\ & + \ell_h \|F\|_{(W_1^1(\Omega))'} \Big) \\ & + \ell_h H^k \|F\|_{(W_1^2(\Omega))'} \Big), \end{aligned} \tag{3.9}$$

where

$$w = (h^{r-2-k} H^{s-(r-2-k)})^{1/s} \tag{3.10}$$

and C depends on s.

Corollary 3.2. *If $k \geq r-2$ is an integer and $0 \leq s \leq r-2$ then*

$$\begin{aligned}\|u - u_h\|_{W_\infty^{-k}(U,\Omega)} \leq CH^{k-(r-2)}\Bigg(&h^{r-1}\ell_{s=r-2,h,h/H}\Big(\inf_{\chi\in S_h}\|u-\chi\|_{W_\infty^1(\Omega),U,H,s} \\ &+ \ell_h\|F\|_{(W_1^1(\Omega))'}\Big) \\ &+ \ell_h H^k\|F\|_{(W_1^2(\Omega))'}\Bigg),\end{aligned} \tag{3.11}$$

where C depends on k and s.

3.2 Motivation

In this section, we motivate u, u_h, F, and their relationship in Equation 3.8.

Define the differential operator L on functions $v : \Omega \to \mathbb{R}$ by

$$Lv = -\sum_{i,j=1}^{N} D_i(a_{i,j}D_jv) + \sum_{i=1}^{N} b_iD_iv + cv. \tag{3.12}$$

The corresponding co-normal derivative operator B is defined on functions $v : \bar{\Omega} \to \mathbb{R}$ by

$$Bv = \sum_{i,j=1}^{N} a_{i,j}(\nu_\Omega)_jD_iv. \tag{3.13}$$

We typically think of $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ as the solution of the classical homogeneous Neumann problem

$$\begin{aligned} Lu &= f \quad \text{on } \Omega \\ Bu &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.14}$$

where $f \in C^0(\Omega)$ is given.

We may also think of $u \in W_2^1(\Omega)$ as the solution of the weak problem

$$A(u,v) = \int_\Omega fv \tag{3.15}$$

for all $v \in W_2^1(\Omega)$, where $f \in L_2(\Omega)$ is given. By integration by parts, it is easily seen that, if u is a solution of the classical problem, then it is a solution of the weak problem. Notice that the weak problem admits solutions with less regularity than the classical problem.

In general, it is not feasible to find an explicit formula for u, so we resort to numerical methods to approximate it. We think of S_h as an abstract finite element space and I_h as its interpolant. Assumptions 1, 2, and 3 are standard approximation, inverse, and superapproximation properties, respectively, and are essentially the same as those in [19, Section 1(B)]. These are satisfied by many commonly-used finite element schemes. The finite element approximation of the solution of Equation 3.15 is the unique solution $u_h \in S_h$ of the finite-dimensional linear system

$$A(u_h, \chi) = \int_\Omega f\chi \tag{3.16}$$

for all $\chi \in S_h$. From Equations 3.15 and 3.16, we have that

$$A(u - u_h, \chi) = 0 \tag{3.17}$$

for all $\chi \in S_h$. This is the $F = 0$ case of Equation 3.8.

It is advantageous to study the more general Equation 3.8 instead of Equation 3.17 because nonvanishing F arise in many applications. For instance, numerical quadrature must often be used to approximate the integrals on both sides of Equation 3.16. This issue is discussed in [25, Section 5], [14, Theorem 1.4], and Chapter 5. In [9, Theorems 3.1 and 5.1], solutions of nonlinear problems are treated as perturbations of solutions of linear ones.

Theorem 3.1 and Corollary 3.2 give estimates for general L_∞-based negative norms of the finite element error $u - u_h$ on arbitrary sets. Corollary 3.2 is an immediate consequence of Poincaré's inequality and Theorem 3.1.

3.3 Relationship to Prior Work

3.3.1 Improvement Over Announced Results

Theorem 3.1 was initially announced in [22, Theorem 3]. Several improvements have been made to this. First, U is now allowed to be any arbitrary subset of $\mathbb{R}^N$ instead of just a ball centred at a point in Ω. Second, the more general definition of the negative norm is used. Third, the initial announcement covered the cases $0 \le s \le r-2-k$ and had a logarithmic factor of $\ell_h \ell_{h,s=r-2-k}$ multiplying the first term on the right side of Equation 3.9. Now, the $s = r-2-k$ case has the much smaller logarithmic factor of $\ell_{h/H}$ here instead, and the cases $0 \le s < r-2-k$ are of no interest. New results are obtained in the cases $r-2-k < s \le r-2$, although we must take our weight parameter $w \ge h$. Fourth, we now allow for the possibility that $F \ne 0$.

3.3.2 An Extrapolation of Positive Norm Results

A pointwise estimate for the difference between the true solution and the finite element solution is given in [19, Theorem 2.2]. If $x \in \bar{\Omega}$ and $0 \le s \le r-2$ then

$$
\begin{aligned}
|(u-u_h)(x)| \le C\Bigg(& h\ell_{s=r-2,h}\Big(\inf_{\chi \in S_h} \|u-\chi\|_{W^1_\infty(\Omega),\{x\},h,s} \\
& + \|F\|_{(W^1_1(\Omega),\{x\},h,-s)'}\Big) \\
& + \ell_h \|F\|_{(W^2_1(\Omega))'}\Bigg),
\end{aligned}
\tag{3.18}
$$

where C depends on s. The $\mathring{W}^1_1(\Omega)$ and $\mathring{W}^2_1(\Omega)$ in [19, Equations 2.5 and 2.6] should read $W^1_1(\Omega)$ and $W^2_1(\Omega)$, respectively.

An estimate for the maximum difference between the first derivatives of the true solution and the finite element solution in small neighbourhoods is given in

[19, Theorem 3.2]. If $x \in \bar{\Omega}$ and $0 \leq s \leq r-1$ then

$$\begin{aligned} \|u - u_h\|_{W^1_\infty(B_h(x)\cap\Omega)} \leq C\Big(&\ell_{s=r-1,h} \inf_{\chi\in S_h} \|u-\chi\|_{W^1_\infty(\Omega),\{x\},h,s} \\ &+ \ell_h \|F\|_{(W^1_1(\Omega))'}\Big), \end{aligned} \tag{3.19}$$

where C depends on s. The $\mathring{W}^1_1(\Omega)$ in [19, Theorem 3.2] should read $W^1_1(\Omega)$.

Notice that the estimates for the error in the positive norms of Equations 3.18 and 3.19 involve single points or small neighbourhoods on the left side and distances to the point or to the centre of the neighbourhood on the right side. In contrast, the estimates for the error in the negative norms of Theorem 3.1 involve an arbitrary set on the left side and the distance to that set on the right side. We now put the positive norm estimates in this form. This is easily done using the observation that, if $x \in U \cap \Omega$ then $\sigma_{\{x\},h} \leq \sigma_{U,h}$. If $0 \leq s \leq r-2$ then, by Equation 3.18,

$$\begin{aligned} \|u - u_h\|_{L_\infty(U\cap\Omega)} \leq C\Bigg(&h\ell_{s=r-2,h}\Big(\inf_{\chi\in S_h} \|u-\chi\|_{W^1_\infty(\Omega),U,h,s} \\ &+ \|F\|_{(W^1_1(\Omega),U,h,-s)'}\Big) \\ &+ \ell_h \|F\|_{(W^2_1(\Omega))'}\Bigg), \end{aligned} \tag{3.20}$$

where C depends on s. If $0 \leq s \leq r-1$ then, by Equation 3.19,

$$\begin{aligned} \|u - u_h\|_{W^1_\infty(U\cap\Omega)} \leq C\Big(&\ell_{s=r-1,h} \inf_{\chi\in S_h} \|u-\chi\|_{W^1_\infty(\Omega),U,h,s} \\ &+ \ell_h \|F\|_{(W^1_1(\Omega))'}\Big), \end{aligned} \tag{3.21}$$

where C depends on s.

At this point, we can make a sensible comparison of the positive norm error estimates of Equations 3.20 and 3.21 with the negative norm error estimates of Theorem 3.1. If $F = 0$, $U \subset \Omega$, and $k \in -(r-2) : 1$ then

$$\|u - u_h\|_{W^k_\infty(U)} \leq Ch^{1-k}\ell_{k\geq 0,h,h/H} \inf_{\chi\in S_h} \|u-\chi\|_{W^1_\infty(\Omega),U,h,r-2+k}. \tag{3.22}$$

Notice that the weight power is chosen to be $r-2+k$. This is precisely the condition under which the logarithmic factor ℓ_h is present for the positive norms in Equations 3.20 and 3.21. By Equation 3.10, it is also the condition under which the weight parameter is h for the negative norms.

3.3.3 A Sharpening of Previous W_∞^{-1} Results

Estimates for the error in the W_∞^{-1} norm are given in [9, Lemma 5.4]. The proof of these results uses the finite element space inverse assumption in a way that prevents an extension to W_∞^{-k} estimates for any integer $k>1$. Here we avoid this limitation by using the inverse property of mollifiers instead of the finite element space inverse assumption.

If $U \subset \Omega$, $\delta > 0$, $r \geq 3$, and $0 \leq s \leq r-2$ then, by [9, Equation 5.4],

$$\begin{aligned} \|u-u_h\|_{W_\infty^{-1}(U)} \leq C\Bigg(& h^2\Big(\big(\frac{H}{h}\big)^\delta + \ell_{s=r-2,h}\Big)\Big(\inf_{\chi\in S_h} \|u-\chi\|_{W_\infty^1(\Omega),U,H,s} \\ & + \ell_h \|F\|_{(W_1^1(\Omega))'}\Big) \\ & + H\ell_H\|F\|_{(W_1^2(\Omega))'}\Bigg), \end{aligned} \tag{3.23}$$

where C depends on s and δ. One difference between this and Equation 3.9 is that the first term on the right side of the former has the factor $(\frac{H}{h})^\delta + \ell_{s=r-2,h}$, whereas that of the latter has the factor $\ell_{s=r-2,h,H/h}$. In [9, Remark 5.2], it is noted that the $(\frac{H}{h})^\delta$ could likely be improved to a logarithmic factor, presumably $\ell_{H/h}$. This would be the case if a certain dependence on p of L_p-based estimates of the solutions of second-order partial differential equations with first-order homogeneous boundary conditions could be established. The desired dependence is given by Theorem 2.3. With this, the factor in question in Equation 3.23 matches that of Equation 3.9.

The other difference between Equation 3.23 and Equation 3.9 is that the first term on the right side of the former has weight parameter H, whereas that of the latter has weight parameter w, which we can take to be

$$w = \begin{cases} h, & \text{if } 0 \le s \le r-3 \\ (h^{r-3} H^{s-(r-3)})^{1/s}, & \text{if } r-3 \le s \le r-2. \end{cases} \tag{3.24}$$

If $h < H$ then we always have $w < H$, which makes Theorem 3.1 sharper. The crucial tool in proving these sharper results is Theorem 2.4. This gives local L_1-based estimates for solutions of homogeneous second-order partial differential equations satisfying homogeneous first-order boundary conditions.

If $r = 2$ and $U \subset \Omega$ then, by [9, Equation 5.3],

$$\begin{aligned} \|u - u_h\|_{W_\infty^{-1}(U)} \le CH\ell_h \Bigg(h \Big(& \inf_{\chi \in S_h} \|u - \chi\|_{W_\infty^1(\Omega)} \\ & + \|F\|_{(W_1^1(\Omega))'} \Big) \\ & + \|F\|_{(W_1^2(\Omega))'} \Bigg). \end{aligned} \tag{3.25}$$

This agrees exactly with Corollary 3.2.

3.3.4 A Trivial Estimate

We consider a trivial estimate obtained using Poincaré's inequality and the pointwise error estimate of [19, Theorem 2.2].

If $k \ge 1$ is an integer and $0 \le s \le r-2$ then, by Equation 3.20 and Poincaré's inequality,

$$\begin{aligned} \|u - u_h\|_{W_\infty^{-k}(U,\Omega)} \le CH^k \Bigg(h\ell_{s=r-2,h} \Big(& \inf_{\chi \in S_h} \|u - \chi\|_{W_\infty^1(\Omega),U,h,s} \\ & + \|F\|_{(W_1^1(\Omega),U,h,-s)'} \Big) \\ & + \ell_h \|F\|_{(W_1^2(\Omega))'} \Bigg), \end{aligned} \tag{3.26}$$

where C depends on k and s.

First we compare the cases $k \le r-2$ of Equation 3.26 with Equation 3.9. Equation 3.26 has the advantage over Equation 3.9 that the weight parameter appearing in the first term on the right side on the former is h, which is smaller, in general, than the w of the latter. Equation 3.9 has the advantage over Equation 3.26 that the first term on the right side of the former is multiplied by h^k, which is smaller, in general, than the H^k of the latter. Disregarding logarithmic factors, the first term on the right side of Equation 3.9 is smaller than that of Equation 3.26. This is because, for $x \in \Omega$,

$$\begin{aligned} h^k \sigma_{U,w}^s(x) &= h^k \frac{w^s}{(w + \operatorname{dist}(x,U))^s} \\ &= \left(\frac{h}{H}\right)^{r-2-s} H^k \frac{h^s}{(w + \operatorname{dist}(x,U))^s} \\ &\le H^k \frac{h^s}{(h + \operatorname{dist}(x,U))^s} \\ &= H^k \sigma_{U,h}^s(x), \end{aligned} \tag{3.27}$$

so $h^{k+1}\sigma_{U,w}^s \le H^k h \sigma_{U,h}^s$.

Next we compare the cases $k \ge r-2$ of Equation 3.26 with Equation 3.11. Again, disregarding logarithmic factors, the first term on the right side of Equation 3.11 is smaller than that of Equation 3.26. This is because, for $x \in \Omega$,

$$\begin{aligned} H^{k-(r-2)} h^{r-2} \sigma_{U,H}^s(x) &= H^{k-(r-2)} h^{r-2} \frac{H^s}{(H + \operatorname{dist}(x,U))^s} \\ &= \left(\frac{h}{H}\right)^{r-2-s} H^k \frac{h^s}{(H + \operatorname{dist}(x,U))^s} \\ &\le H^k \frac{h^s}{(h + \operatorname{dist}(x,U))^s} \\ &= H^k \sigma_{U,h}^s(x), \end{aligned} \tag{3.28}$$

so $H^{k-(r-2)} h^{r-1} \sigma_{U,H}^s \le H^k h \sigma_{U,h}^s$.

The second term on the right side of Equation 3.26 has no straightforward comparison to that of Equations 3.9 and 3.11. The third terms, however, are all the same.

3.4 Proof of Results

In this section, we prove Theorem 3.1.

By the general definition of the negative norm,

$$\|u-u_h\|_{W_\infty^{-k}(U,\Omega)} = \sup_{\substack{\phi\in C_0^\infty(U)\\ \|\phi\|_{W_1^k(U)}=1}} \Big|\int_{U\cap\Omega}(u-u_h)\phi\Big|. \tag{3.29}$$

Let $\phi \in C_0^\infty(U)$ have $\|\phi\|_{W_1^k(U)} = 1$. By Theorem 3.3, there exists an open subset V of $\mathbb{R}^N$ and some $\psi \in C_0^\infty(V)$ such that $U \subset V$, $\operatorname{dist}(U, \partial V) \le Ch$, and, if $1 \le p \le \infty$ then

$$\|\phi-\psi\|_{L_p(V)} \le Ch^k|\phi|_{W_p^k(U)} \tag{3.30}$$

and

$$\|\psi\|_{W_p^k(V)} \le Ch^{-N/p'}\|\phi\|_{W_1^k(U)}. \tag{3.31}$$

By Equation 3.30, Poincaré's inequality, and Equation 3.31, we see that, if $1 \le p \le \infty$, then

$$\begin{aligned}
\|\psi\|_{L_p(V)} &\le \|\phi-\psi\|_{L_p(V)} + \|\phi\|_{L_p(V)} \\
&\le C(h^k+H^k)|\phi|_{W_p^k(V)} \\
&\le CH^kh^{-N/p'}\|\phi\|_{W_1^k(U)} \\
&= CH^kh^{-N/p'}.
\end{aligned} \tag{3.32}$$

Define the bilinear form $A^\dagger$ on functions $v, w : \Omega \to \mathbb{R}$ by

$$A^\dagger(v,w) = A(w,v) = \int_\Omega\Big(\sum_{i,j=1}^N a_{i,j}D_jvD_iw + \sum_{i=1}^N b_ivD_iw + cvw\Big). \tag{3.33}$$

Notice that $A^\dagger$ has the same coercivity and ellipticity constants as A. Define the differential operator $L^\dagger$ on functions $v : \Omega \to \mathbb{R}$ by

$$L^\dagger v = -\sum_{i,j=1}^N D_j(a_{i,j}D_iv) - \sum_{i=1}^N D_i(b_iv) + cv, \tag{3.34}$$

and define the boundary differential operator $B^\dagger$ on functions $v : \bar{\Omega} \to \mathbb{R}$ by

$$B^\dagger v = \sum_{i,j=1}^{N} a_{i,j}(\nu_\Omega)_j D_i v + \sum_{i=1}^{N} b_i(\nu_\Omega)_i v. \tag{3.35}$$

Since $A^\dagger$ is coercive over $W_2^1(\Omega)$, we know by the Lax-Milgram theorem that there exists a unique $v \in W_2^1(\Omega)$ such that

$$A^\dagger(v, w) = \int_\Omega \psi w \tag{3.36}$$

for all $w \in W_1^2(\Omega)$. Although regularity of weak solutions of the Dirichlet problem is established in [10, Section 6.3, Theorems 1 and 4], the same arguments, with very slight and obvious modification, can be used to show that $v \in W_2^2(\Omega)$ here. By integration by parts, it can now be seen that

$$\begin{aligned} L^\dagger v &= \psi \quad \text{on } \Omega \\ B^\dagger v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.37}$$

By [2, Theorem 15.2], $v \in W_2^r(\Omega)$.

By Equations 3.36 and 3.33,

$$\int_\Omega (u - u_h)\psi = A(u - u_h, v). \tag{3.38}$$

By Equation 3.8,

$$A(u - u_h, v) = A(u - u_h, v - I_h v) + F(I_h v). \tag{3.39}$$

Since F is linear,

$$F(I_h v) = F(v) - F(v - I_h v). \tag{3.40}$$

Putting together Equations 3.38, 3.39, and 3.40, we find that

$$\begin{aligned} |\int_{U\cap\Omega} (u - u_h)\phi| &= |\int_{V\cap\Omega} (u - u_h)\phi| \\ &\leq |\int_{V\cap\Omega} (u - u_h)(\phi - \psi)| + |\int_{V\cap\Omega} (u - u_h)\psi| \\ &\leq \|u - u_h\|_{L_\infty(V\cap\Omega)} \|\phi - \psi\|_{L_1(V\cap\Omega)} \\ &\qquad + |A(u - u_h, v - I_h v)| + |F(v - I_h v)| + |F(v)|. \end{aligned} \tag{3.41}$$

To prove Theorem 3.1, it will suffice to show that each of the four terms on the right side of this equation are bounded by the right side of Equation 3.9. The inequalities

$$\|v - I_h v\|_{W_1^1(\Omega),U,w,-s} \leq Ch^{k+1}\ell_{s=r-2,h,h/H} \tag{3.42}$$

and

$$\|v\|_{W_1^2(\Omega)} \leq CH^k \ell_h \tag{3.43}$$

are central to this endeavour. Before proving these two inequalities, we see how Theorem 3.1 follows from them.

By Equation 3.30,

$$\begin{aligned} \|\phi - \psi\|_{L_1(V\cap\Omega)} &\leq Ch^k |\phi|_{W_1^k(U)} \\ &= Ch^k. \end{aligned} \tag{3.44}$$

If $x \in V \cap \Omega$ then $\operatorname{dist}(x, U) \leq Ch$, from which it is readily observed that $\sigma_{\{x\},h} \leq C\sigma_{U,h}$. Therefore, by Equation 3.18,

$$\begin{aligned} \|u - u_h\|_{L_\infty(V\cap\Omega)} \leq Ch^k \Bigg(& h\ell_{s=r-2,h}\Big(\inf_{\chi\in S_h} \|u-\chi\|_{W_\infty^1(\Omega),U,h,s} \\ & + \|F\|_{(W_1^1(\Omega),U,h,-s)'}\Big) \\ & + \ell_h \|F\|_{(W_1^2(\Omega))'}\Bigg). \end{aligned} \tag{3.45}$$

Multiplying both sides of Equation 3.19 by $\sigma_{U,w}^s(x)$, using the multiplicative property of weights, and taking the maximum over all $x \in \Omega$ gives

$$\begin{aligned} \|u - u_h\|_{W_\infty^1(\Omega),U,w,s} \leq C\Big(& \ell_{s=r-1,h} \inf_{\chi\in S_h} \|u - \chi\|_{W_\infty^1(\Omega),U,w,s} \\ & + \ell_h \|F\|_{(W_1^1(\Omega))'}\Big). \end{aligned} \tag{3.46}$$

Using Equations 3.46 and 3.42,

$$\begin{aligned} |A(u-u_h, v - I_h v)| &\leq \|u-u_h\|_{W_\infty^1(\Omega),U,w,s} \|v - I_h v\|_{W_1^1(\Omega),U,w,-s} \\ &\leq Ch^{k+1}\ell_{s=r-2,h,h/H}\Big(\inf_{\chi\in S_h}\|u-\chi\|_{W_\infty^1(\Omega),U,w,s} \\ &\qquad + \ell_h\|F\|_{(W_1^1(\Omega))'}\Big). \end{aligned} \tag{3.47}$$

Using Equation 3.42,

$$\begin{aligned} |F(v - I_h v)| &\le \|v - I_h v\|_{W_1^1(\Omega),U,w,-s} \|F\|_{(W_1^1(\Omega),U,w,-s)'} \\ &\le C h^{k+1} \ell_{s=r-2,h,h/H} \|F\|_{(W_1^1(\Omega),U,w,-s)'}. \end{aligned} \tag{3.48}$$

Using Equation 3.43,

$$\begin{aligned} |F(v)| &\le \|v\|_{W_1^2(\Omega)} \|F\|_{(W_1^2(\Omega))'} \\ &\le C H^k \ell_h \|F\|_{(W_1^2(\Omega))'}. \end{aligned} \tag{3.49}$$

By Equations 3.44, 3.45, 3.47, 3.48, and 3.49, we see that all four terms on the right side of Equation 3.41 are bounded by the right side of Equation 3.9. Therefore, it only remains to prove Equations 3.42 and 3.43.

If $U \cap \Omega = \emptyset$ then the left side of Equation 3.9 is zero, and Equation 3.9 is trivial. Therefore, we can assume that $U \cap \Omega \ne \emptyset$. Let $R = \operatorname{diam}(\Omega)$ and notice that, if $x \in \Omega$ then $\operatorname{dist}(x, U) < R$.

We first define a sequence of annuli around U. For $i \ge 0$ an integer, let $d_i = e^i$,

$$U_i = \{x \in \mathbb{R}^N : d_i < \operatorname{dist}(x, U) < d_{i+1}\}, \tag{3.50}$$

$$U_i' = U_{i-1} \cup U_i \cup U_{i+1}, \tag{3.51}$$

and

$$U_i'' = U_{i-1}' \cup U_i' \cup U_{i+1}'. \tag{3.52}$$

Notice that $U_i \subset U_i' \subset U_i''$, $\operatorname{dist}(U_i, \partial U_i') = e^{-1} d_i$, and $\operatorname{dist}(U_i', \partial U_i'') = e^{-2} d_i$.

Let I_{ext} be the smallest integer such that $d_{I_{\text{ext}}+1} \ge R$, let I_{int} be the smallest integer such that $d_{I_{\text{int}}} \ge w$ and $d_{I_{\text{int}}-1} \ge C_{\text{sep}} h$, and let $I_{\text{mid}} \ge I_{\text{int}} + 1$ be the smallest integer such that $d_{I_{\text{mid}}} \ge H$ and $d_{I_{\text{mid}}-1} \ge \operatorname{dist}(U, \partial V)$. Notice that $d_{I_{\text{int}}}$ is bounded above and below by Cw, $d_{I_{\text{mid}}}$ is bounded above and below by CH, and $d_{I_{\text{ext}}} \le C$. If $i \ge I_{\text{int}}$ then $\operatorname{dist}(U_i, \partial U_i') \ge C_{\text{sep}} h$. If $i \ge I_{\text{mid}} + 1$ then $\operatorname{dist}(U_i', V) \ge d_{i-1} - d_{I_{\text{mid}}-1} \ge (e^{-1} - e^{-2}) d_i$. If $i > I_{\text{ext}}$ then $U_i \cap \Omega = \emptyset$.

Define

$$U_{\text{int}} = \{x \in \mathbb{R}^N : \text{dist}(x, U) < d_{I_{\text{int}}}\}, \tag{3.53}$$

$$U'_{\text{int}} = \{x \in \mathbb{R}^N : \text{dist}(x, U) < d_{I_{\text{int}}+1}\}, \tag{3.54}$$

and

$$U_{\text{mid}} = \{x \in \mathbb{R}^N : \text{dist}(x, U) < d_{I_{\text{mid}}}\}. \tag{3.55}$$

Notice that $U_{\text{int}} \subset U'_{\text{int}} \subset U_{\text{mid}}$ and $\text{dist}(U_{\text{int}}, \partial U'_{\text{int}}) \geq (e-1)d_{I_{\text{int}}} \geq C_{\text{sep}}h$.

Roughly speaking, the set U_{int} includes points which are at a distance up to w from U, the set U_{mid} includes points which are at a distance up to H from U, and the annulus U_i contains points that are at a distance between d_i and d_{i+1} from U. The sets $\bar{U}_i$ for $i \in I_{\text{int}} : I_{\text{ext}}$, along with $\bar{U}_{\text{int}}$, cover Ω.

First we show Equation 3.42. Let $p = \ell_{h/H}$. By the measure inequality,

$$\|v\|_{W_1^{k+2}(U_{\text{mid}} \cap \Omega)} \leq CH^{N/p} \|v\|_{W_{p'}^{k+2}(U_{\text{mid}} \cap \Omega)}, \tag{3.56}$$

and by Theorem 2.3,

$$\|v\|_{W_{p'}^{k+2}(\Omega)} \leq Cp \|\psi\|_{W_{p'}^{k}(\Omega)}. \tag{3.57}$$

By Equations 3.56, 3.57, and 3.31,

$$\begin{aligned} \|v\|_{W_1^{k+2}(U_{\text{mid}} \cap \Omega)} &\leq C \left(\frac{H}{h}\right)^{N/p} p \|\phi\|_{W_1^k(U)} \\ &\leq C\ell_{h/H}. \end{aligned} \tag{3.58}$$

On U_{int}, $\sigma_{U,w}^{-1} \leq C$, so

$$\|v - I_h v\|_{W_1^1(U_{\text{int}} \cap \Omega), U, w, -s} \leq C \|v - I_h v\|_{W_1^1(U_{\text{int}} \cap \Omega)}. \tag{3.59}$$

By the finite element space approximation assumption,

$$\|v - I_h v\|_{W_1^1(U_{\text{int}} \cap \Omega)} \leq Ch^{k+1} \|v\|_{W_1^{k+2}(U'_{\text{int}} \cap \Omega)}. \tag{3.60}$$

Combining Equations 3.59, 3.60, and 3.58, we see that

$$\|v - I_h v\|_{W_1^1(U_{\text{int}} \cap \Omega), U, w, -s} \leq Ch^{k+1} \ell_{h/H}. \tag{3.61}$$

Next we make some general observations about the annuli. Suppose that $i \in I_{\text{int}} : I_{\text{ext}}$. On U_i, $\sigma_{U,w}^{-1} \leq \frac{w+ed_i}{w} \leq C\frac{d_i}{w}$, so

$$\|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} \leq C\Big(\frac{d_i}{w}\Big)^s \|v - I_h v\|_{W_1^1(U_i\cap\Omega)}. \tag{3.62}$$

By the finite element space approximation assumption,

$$\|v - I_h v\|_{W_1^1(U_i\cap\Omega)} \leq Ch^{r-1}\|v\|_{W_1^r(U_i'\cap\Omega)}. \tag{3.63}$$

Putting together Equations 3.62 and 3.63,

$$\|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} \leq C\Big(\frac{d_i}{w}\Big)^s h^{r-1}\|v\|_{W_1^r(U_i'\cap\Omega)}. \tag{3.64}$$

Now we look at the small annuli. These are roughly at distances between w and H from U. If $i \in I_{\text{int}} : I_{\text{mid}}$ then, by Theorem 2.4,

$$\|v\|_{W_1^r(U_i'\cap\Omega)} \leq Cd_i^{-(r-2-k)}\|v\|_{W_1^{k+2}(U_i''\cap\Omega)}. \tag{3.65}$$

Now we sum up the contributions from all the small annuli. By Equations 3.64, 3.65, 3.10, and 3.58,

$$\begin{aligned}\sum_{i=I_{\text{int}}}^{I_{\text{mid}}} \|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} &\leq Ch^{r-1}w^{-s}\sum_{i=I_{\text{int}}}^{I_{\text{mid}}} d_i^{s-(r-2-k)}\|v\|_{W_1^{k+2}(U_i''\cap\Omega)} \\ &\leq Ch^{r-1}w^{-s}H^{s-(r-2-k)}\|v\|_{W_1^{k+2}(U_{\text{mid}}\cap\Omega)} \\ &\leq Ch^{k+1}\ell_{h/H}.\end{aligned} \tag{3.66}$$

Now we look at the large annuli. These are roughly at distances greater than H from U. If $i \in I_{\text{mid}} + 1 : I_{\text{ext}}$ then, by Theorem 2.5,

$$\|v\|_{W_1^r(U_i'\cap\Omega)} \leq Cd_i^{-(r-2)}\|\psi\|_{L_1(V\cap\Omega)}. \tag{3.67}$$

If $s < r - 2$ then, by the geometric series formula,

$$\begin{aligned}\sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} d_i^{-(r-2-s)} &\leq Cd_{I_{\text{mid}}}^{-(r-2-s)} \\ &\leq CH^{-(r-2-s)}.\end{aligned} \tag{3.68}$$

If $s = r - 2$ then

$$\sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} d_i^{-(r-2-s)} = I_{\text{ext}} - I_{\text{mid}} \\ \leq C + \log \frac{1}{H} \\ \leq C\ell_H. \tag{3.69}$$

Putting together Equations 3.68 and 3.69,

$$\sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} d_i^{-(r-2-s)} \leq CH^{-(r-2-s)}\ell_{s=r-2,H}. \tag{3.70}$$

Now we sum up the contributions from all the large annuli. By Equations 3.64, 3.67, 3.32, 3.70, and 3.10,

$$\begin{aligned} \sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} \|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} &\leq Ch^{r-1}w^{-s}H^k \sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} d_i^{-(r-2-s)} \\ &\leq Ch^{r-1}w^{-s}H^{s-(r-2-k)}\ell_{s=r-2,H} \\ &= Ch^{k+1}\ell_{s=r-2,H}. \end{aligned} \tag{3.71}$$

Finally we add up the contributions from the innermost domain, the small annuli, and the large annuli. By Equations 3.61, 3.66 and 3.71,

$$\begin{aligned} \|v - I_h v\|_{W_1^1(\Omega),U,w,-s} &= \|v - I_h v\|_{W_1^1(U_{\text{int}}\cap\Omega),U,w,-s} \\ &\quad + \sum_{i=I_{\text{int}}}^{I_{\text{mid}}} \|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} \\ &\quad + \sum_{i=I_{\text{mid}}+1}^{I_{\text{ext}}} \|v - I_h v\|_{W_1^1(U_i\cap\Omega),U,w,-s} \\ &\leq Ch^{k+1}(\ell_{h/H} + \ell_{s=r-2,H}) \\ &\leq Ch^{k+1}\ell_{s=r-2,h,h/H}, \end{aligned} \tag{3.72}$$

which establishes Equation 3.42.

It remains only to show Equation 3.43. Let $p = \ell_h$. By the measure inequality,

$$\|v\|_{W_1^2(\Omega)} \leq C\|v\|_{W_{p'}^2(\Omega)}, \tag{3.73}$$

and by Theorem 2.3,

$$\|v\|_{W^2_{p'}(\Omega)} \le Cp\|\psi\|_{L_{p'}(\Omega)}. \tag{3.74}$$

By Equations 3.73, 3.74, and 3.32,

$$\begin{aligned} \|v\|_{W^2_1(\Omega)} &\le CH^k h^{N/p} p \\ &\le CH^k \ell_h, \end{aligned} \tag{3.75}$$

which establishes Equation 3.43.

3.5 Appendix: Mollifiers

The typical application of mollification is to extend a result for smooth functions to nonsmooth functions. This is done by applying the result to a sequence of mollifications of a nonsmooth function. In this sort of an argument, the fact that the sequence converges is crucial, but the rate of convergence is irrelevant.

In this appendix, we begin with infinitely differentiable functions and approximate them with their mollifications. Our result is framed in terms of simultaneous approximation and inverse properties in various Sobolev space seminorms.

Theorem 3.3. *Suppose that $m \ge 0$ is an integer, $d > 0$, U is an open subset of $\mathbb{R}^N$, $u \in C_0^\infty(U)$, and $V = \{x \in \mathbb{R}^N : \operatorname{dist}(x, U) < d\sqrt{N}(m+1)\}$. Then there exists some $v \in C_0^\infty(V)$ with the following properties.*

1. *If $1 \le p \le \infty$, $k \ge 0$ is an integer, and $\ell \in k : k + 2m + 2$ then*

$$|u - v|_{W^k_p(V)} \le Cd^{\ell-k}|u|_{W^\ell_p(U)}, \tag{3.76}$$

where C depends on N and m.

2. *If $1 \le q \le p \le \infty$, $k \ge 0$ is an integer, and $\ell \in 0 : k$ then*

$$|v|_{W^k_p(V)} \le Cd^{-N(1/p-1/q)-(k-\ell)}|u|_{W^\ell_q(U)}, \tag{3.77}$$

where C depends on N, m, and k.

Let $J \in C_0^\infty(C_{1/2}(0))$ be an even function such that $\int_{\mathbb{R}} J = 1$.

Proposition 3.4. *There exist $c_0, \ldots, c_m \in \mathbb{R}$ such that, for all $i \in 0 : m$,*

$$\frac{1}{2}\sum_{j=0}^{m} \int_{\mathbb{R}} c_j (J(x-j) + J(x+j)) x^{2i}\, dx = \delta_{i,0}. \tag{3.78}$$

Proof. We exhibit $c \in \mathbb{R}^{0:m}$ as the solution of a nonsingular linear system of equations.

If $i \geq 0$ is an integer then, using the change of variables formula and the fact that J is even,

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{m} \int_{\mathbb{R}} c_j (J(x-j) + J(x+j)) x^{2i}\, dx \\
&= \frac{1}{2}\sum_{j=0}^{m} c_j \left(\int_{\mathbb{R}} J(x-j) x^{2i}\, dx + \int_{\mathbb{R}} J(-x-j) x^{2i}\, dx \right) \\
&= \frac{1}{2}\sum_{j=0}^{m} c_j \left(\int_{\mathbb{R}} J(x)(x+j)^{2i}\, dx + \int_{\mathbb{R}} J(x)(-x-j)^{2i}\, dx \right) \\
&= \sum_{j=0}^{m} c_j \int_{\mathbb{R}} J(x)(x+j)^{2i}\, dx.
\end{aligned} \tag{3.79}$$

If $x \in \mathbb{R}$, $j \in 0 : m$, and $i \geq 0$ is an integer, then, by the binomial theorem,

$$(x+j)^{2i} = \sum_{k=0}^{2i} \binom{2i}{k} j^k x^{2i-k}. \tag{3.80}$$

If i and k are integers and k is odd then $2i-k$ is odd so $x \mapsto x^{2i-k}$ is odd, and therefore, since J is even, $\int_{\mathbb{R}} J(x) x^{2i-k}\, dx = 0$. Thus, by Equation 3.80, if $j \in 0 : m$ and $i \geq 0$ is an integer then

$$\int_{\mathbb{R}} J(x)(x+j)^{2i}\, dx = \sum_{k=0}^{i} \binom{2i}{2k} j^{2k} \int_{\mathbb{R}} J(x) x^{2(i-k)}\, dx. \tag{3.81}$$

For integers $\ell \geq 0$, define

$$a_\ell = \int_{\mathbb{R}} J(x) x^{2\ell}\, dx. \tag{3.82}$$

Notice that $a_0 = \int_{\mathbb{R}} J = 1$. Define $A \in \mathbb{R}^{(0:m)\times(0:m)}$ by

$$A_{i,j} = \sum_{k=0}^{i} \binom{2i}{2k} a_{i-k} j^{2k}. \tag{3.83}$$

Also define $b \in \mathbb{R}^{0:m}$ by

$$b_i = \delta_{i,0}. \tag{3.84}$$

By Equations 3.79, 3.81, 3.82, 3.83, and 3.84, we see that the claim of the proposition is that there exists a solution c of $Ac = b$. We now show that A is nonsingular.

Define $L, V \in \mathbb{R}^{(0:m)\times(0:m)}$ by

$$L_{i,k} = \begin{cases} \binom{2i}{2k} a_{i-k}, & \text{if } k \leq i \\ 0, & \text{otherwise} \end{cases} \tag{3.85}$$

and

$$V_{k,j} = j^{2k} = (j^2)^k. \tag{3.86}$$

Since L has 1s along the diagonal and is lower triangular, $\det L = 1$, so L is nonsingular. Since the squares of distinct nonnegative integers are distinct, the Vandermonde matrix V is nonsingular. Now notice that, if $i, j \in 0 : m$ then, by Equations 3.83, 3.85, and 3.86,

$$A_{i,j} = \sum_{k=0}^{m} L_{i,k} V_{k,j}. \tag{3.87}$$

That is, $A = LV$. Since L and V are nonsingular, so is A. $\square$

Now define $J_m : \mathbb{R} \to \mathbb{R}$ by

$$J_m(x) = \frac{1}{2} \sum_{j=0}^{m} c_j (J(x-j) + J(x+j)). \tag{3.88}$$

Since $J \in C_0^\infty(C_{1/2}(0))$, certainly $J_m \in C_0^\infty(C_{m+1}(0))$. If $x \in \mathbb{R}$ then, since J is even,

$$\begin{aligned} J_m(-x) &= \frac{1}{2} \sum_{j=0}^{m} c_j (J(-x-j) + J(-x+j)) \\ &= \frac{1}{2} \sum_{j=0}^{m} c_j (J(x+j) + J(x-j)) \\ &= J_m(x). \end{aligned} \tag{3.89}$$

That is, J_m is even. If i is odd then $x \mapsto x^i$ is odd, so, since J_m is even,

$$\int_{\mathbb{R}} J_m(x) x^i \, dx = 0. \tag{3.90}$$

If $i \in 0 : 2m$ and i is even then, by Equation 3.88 and Proposition 3.4,

$$\int_{\mathbb{R}} J_m(x) x^i \, dx = \delta_{i,0}. \tag{3.91}$$

Combining Equations 3.90 and 3.91, we have that

$$\int_{\mathbb{R}} J_m(x) x^i \, dx = \delta_{i,0} \tag{3.92}$$

for all $i \in 0 : 2m + 1$. Notice that, if $k \geq 0$ is an integer then

$$D^k J_m(x) = \frac{1}{2} \sum_{j=0}^{m} c_j (D^k J(x - j) + D^k J(x + j)), \tag{3.93}$$

so

$$|J_m|_{W_\infty^k(\mathbb{R})} \leq |J|_{W_\infty^k(\mathbb{R})} \sum_{j=0}^{m} |c_j| \leq C, \tag{3.94}$$

where C depends on m and k.

Now define $J_{m,N} : \mathbb{R}^N \to \mathbb{R}$ by

$$J_{m,N}(x) = \prod_{i=1}^{N} J_m(x_i). \tag{3.95}$$

Let $1 \leq p \leq \infty$ and let $k \geq 0$ be an integer. If $|\alpha| = k$ then, for $x \in \mathbb{R}^N$,

$$D^\alpha J_{m,N}(x) = \prod_{i=1}^{N} D^{\alpha_i} J_m(x_i), \tag{3.96}$$

so

$$\begin{aligned} \|D^\alpha J_{m,N}\|_{L_p(\mathbb{R}^N)} &= \prod_{i=1}^{N} \|D^{\alpha_i} J_m\|_{L_p(\mathbb{R})} \\ &\leq \|J_m\|_{W_p^k(\mathbb{R})}^N. \end{aligned} \tag{3.97}$$

Since $\operatorname{supp}(J_m) \subset C_{m+1}(0)$, we see by Equation 3.94 and the measure inequality that

$$|J_{m,N}|_{W_p^k(\mathbb{R}^N)} \leq C, \tag{3.98}$$

where C depends on N, m, and k.

Now define $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi(x) = x/d$ and let $J_{m,N,d} = d^{-N} J_{m,N} \circ \Phi$. That is,

$$J_{m,N,d}(x) = d^{-N} \prod_{i=1}^{N} J_m(x_i/d). \tag{3.99}$$

By Equation 3.98 and a scaling inequality, if $1 \le p \le \infty$ and $k \ge 0$ is an integer then

$$|J_{m,N,d}|_{W_p^k(\mathbb{R}^N)} \le C d^{-N(1-1/p)-k}, \tag{3.100}$$

where C depends on N, m, and k. Since $J_m \in C_0^\infty(C_{m+1}(0))$, certainly $J_{m,N,d} \in C_0^\infty(C_{d(m+1)}(0))$. If $|\alpha| \le 2m+1$ then $\alpha_i \in 0 : 2m+1$ for all $i \in 1 : N$, so, by Equation 3.92 and the change of variables formula,

$$\begin{aligned}
\int_{\mathbb{R}^N} J_{m,N,d}(x) x^\alpha \, dx &= \int_{\mathbb{R}^N} \Big(d^{-N} \prod_{i=1}^{N} J_m(x_i/d) \Big) \Big(\prod_{i=1}^{N} x_i^{\alpha_i} \Big) \, dx \\
&= \prod_{i=1}^{N} \int_{\mathbb{R}} J_m(x_i/d) x_i^{\alpha_i} d^{-1} \, dx_i \\
&= \prod_{i=1}^{N} d^{\alpha_i} \int_{\mathbb{R}} J_m(x_i) x_i^{\alpha_i} \, dx_i \\
&= \prod_{i=1}^{N} d^{\alpha_i} \delta_{\alpha_i,0} \\
&= \delta_{\alpha,0}.
\end{aligned} \tag{3.101}$$

To prove the theorem, we take $v = J_{m,N,d} * u$. If $x \in \operatorname{supp}(u) \subset U$ and $y \in \operatorname{supp}(J_{m,N,d}) \subset C_{d(m+1)}(0)$ then $\operatorname{dist}(x-y, U) \le d\sqrt{N}(m+1)$, so $x - y \in V$. This shows that $\operatorname{supp}(v) \subset V$.

Our first result concerns the approximation property of mollification.

Lemma 3.5. *If $1 \le p \le \infty$ and $k \in 0 : 2m+2$ then*

$$\|J_{m,N,d} * u - u\|_{L_p(\mathbb{R}^N)} \le C d^k |u|_{W_p^k(\mathbb{R}^N)}, \tag{3.102}$$

where C depends on N and m.

Proof. First we consider the $k = 0$ case. Using Young's inequality and Equation 3.100,

$$\begin{aligned}
\|J_{m,N,d} * u - u\|_{L_p(\mathbb{R}^N)} &\leq \|J_{m,N,d} * u\|_{L_p(\mathbb{R}^N)} + \|u\|_{L_p(\mathbb{R}^N)} \\
&\leq \|J_{m,N,d}\|_{L_1(\mathbb{R}^N)}\|u\|_{L_p(\mathbb{R}^N)} + \|u\|_{L_p(\mathbb{R}^N)} \\
&\leq C\|u\|_{L_p(\mathbb{R}^N)},
\end{aligned} \tag{3.103}$$

where C depends on N and m.

From now on, we assume that $k \in 1 : 2m+2$. Observe that, if $x \in \mathbb{R}^N$, then, by Equation 3.101,

$$\begin{aligned}
u(x) &= \sum_{|\alpha| \leq k-1} \frac{1}{\alpha!}(-1)^{|\alpha|} D^\alpha u(x) \delta_{\alpha,0} \\
&= \sum_{|\alpha| \leq k-1} \frac{1}{\alpha!}(-1)^{|\alpha|} D^\alpha u(x) \int_{\mathbb{R}^N} J_{m,N,d}(y) y^\alpha \, dy \\
&= \int_{\mathbb{R}^N} J_{m,N,d}(y) \sum_{|\alpha| \leq k-1} \frac{1}{\alpha!} D^\alpha u(x)(-y)^\alpha \, dy \\
&= \int_{\mathbb{R}^N} J_{m,N,d}(y) T_x^{k-1} u(x-y) \, dy,
\end{aligned} \tag{3.104}$$

so

$$(J_{m,N,d} * u - u)(x) = \int_{\mathbb{R}^N} J_{m,N,d}(y)(u - T_x^{k-1}u)(x-y) \, dy. \tag{3.105}$$

Now define $w : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by $w(x,y) = (u - T_x^{k-1}u)(x-y)$. By Taylor's theorem,

$$w(x,y) = \frac{1}{(k-1)!} \sum_{|\alpha|=k} (-y)^\alpha \int_0^1 D^\alpha u(x - sy) \, ds. \tag{3.106}$$

Since $\operatorname{supp}(J_{m,N,d}) \subset C_{d(m+1)}(0)$, we have by Equations 3.105 and 3.106 that

$$(J_{m,N,d} * u - u)(x) = \int_{C_{d(m+1)}(0)} w(x,y) J_{m,N,d}(y) \, dy. \tag{3.107}$$

Therefore, by Young's inequality,

$$\|J_{m,N,d} * u - u\|_{L_p(\mathbb{R}^N)} \leq \sup_{y \in C_{d(m+1)}(0)} \|w(\cdot, y)\|_{L_p(\mathbb{R}^N)} \|J_{m,N,d}\|_{L_1(C_{d(m+1)}(0))}. \tag{3.108}$$

We now wish to show that, if $y \in C_{d(m+1)}(0)$ then

$$\|w(\cdot, y)\|_{L_p(\mathbb{R}^N)} \leq Cd^k |u|_{W_p^k(\mathbb{R}^N)}, \tag{3.109}$$

where C depends on N and k. Let $y \in C_{d(m+1)}(0)$ and $|\alpha| = k$. Then $|(-y)^\alpha| \leq Cd^k$, where C depends on N, m, and k.

First we consider the $1 \leq p < \infty$ case. If $x \in \mathbb{R}^N$ then, by the measure inequality,

$$|\int_0^1 D^\alpha u(x - sy)\, ds|^p \leq \int_0^1 |D^\alpha u(x - sy)|^p\, ds. \tag{3.110}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} |\int_0^1 D^\alpha u(x - sy)\, ds|^p\, dx &\leq \int_0^1 \int_{\mathbb{R}^N} |D^\alpha u(x - sy)|^p\, dx\, ds \\ &= \int_0^1 \int_{\mathbb{R}^N} |D^\alpha u(z)|^p\, dz\, ds \\ &\leq |u|^p_{W_p^k(\mathbb{R}^N)},\end{aligned} \tag{3.111}$$

from which Equation 3.109 follows.

Now consider the case $p = \infty$. If $x \in \mathbb{R}^N$ then, by the measure inequality,

$$\int_0^1 |D^\alpha u(x - sy)|\, ds \leq |u|_{W_\infty^k(\mathbb{R}^N)}, \tag{3.112}$$

from which Equation 3.109 follows.

Putting together Equations 3.108, 3.109, and 3.100, we see that

$$\|J_{m,N,d} * u - u\|_{L_p(\mathbb{R}^N)} \leq Cd^k |u|_{W_p^k(\mathbb{R}^N)}, \tag{3.113}$$

where C depends on N, m, and k. The dependence on k can be eliminated because the number of possible values of k depends on m. Equations 3.103 and 3.113 give the lemma. $\square$

We now extend our approximation result to higher-order derivatives. The following lemma gives Theorem 3.3, Part 1.

Lemma 3.6. *If $k \geq 0$ is an integer and $\ell \in k : k + 2m + 2$ then*

$$|J_{m,N,d} * u - u|_{W_p^k(\mathbb{R}^N)} \leq Cd^\ell |u|_{W_p^\ell(\mathbb{R}^N)}, \tag{3.114}$$

where C depends on N and m.

Proof. If $|\alpha| = k$ then

$$D^\alpha(J_{m,N,d} * u - u) = J_{m,N,d} * D^\alpha u - D^\alpha u, \tag{3.115}$$

so, since $\ell - k \in 0 : 2m + 2$, we see by Lemma 3.5 that

$$\begin{aligned} \|D^\alpha(J_{m,N,d} * u - u)\|_{L_p(\mathbb{R}^N)} &\leq Cd^{\ell-k}|D^\alpha u|_{W_p^{\ell-k}(\mathbb{R}^N)} \\ &\leq Cd^{\ell-k}|u|_{W_p^\ell(\mathbb{R}^N)}, \end{aligned} \tag{3.116}$$

where C depends on N, m, and $\ell - k$. The dependence on $\ell - k$ can be eliminated because the number of possible values of $\ell - k$ depends on m. The lemma follows by summing this inequality over all $|\alpha| = k$. □

The following lemma gives Theorem 3.3, Part 2.

Lemma 3.7. *If $1 \leq q \leq p \leq \infty$, $k \geq 0$ is an integer, and $\ell \in 0 : k$ then*

$$|J_{m,N,d} * u|_{W_p^k(\mathbb{R}^N)} \leq Cd^{-N(1/p-1/q)-(k-\ell)}|u|_{W_q^\ell(\mathbb{R}^N)}, \tag{3.117}$$

where C depends on N, m, and k.

Proof. Let r be such that $\frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p}$. For $|\alpha| = k$, choose β, γ with $\beta + \gamma = \alpha$, $|\beta| = k - \ell$, and $|\gamma| = \ell$. Then

$$D^\alpha(J_{m,N,d} * u) = D^\beta J_{m,N,d} * D^\gamma u. \tag{3.118}$$

By Young's inequality,

$$\|D^\beta J_{m,N,d} * D^\gamma u\|_{L_p(\mathbb{R}^N)} \leq \|D^\beta J_{m,N,d}\|_{L_r(\mathbb{R}^N)}\|D^\gamma u\|_{L_q(\mathbb{R}^N)}. \tag{3.119}$$

By Equation 3.100,

$$\begin{aligned} \|D^\beta J_{m,N,d}\|_{L_r(\mathbb{R}^N)} &\le |J_{m,N,d}|_{W_r^{k-\ell}(\mathbb{R}^N)} \\ &\le C d^{-N(1-1/r)-(k-\ell)}, \end{aligned} \tag{3.120}$$

where C depends on N, m, and $k-\ell$. The dependence on $k-\ell$ can be replaced by a dependence on k because the number of possible values of $k-\ell$ depends on k. Combining Equations 3.118, 3.119, and 3.120, we see that

$$\|D^\alpha(J_{m,N,d} * u)\|_{L_p(\Omega)} \le C d^{-N(1/q-1/p)-(k-\ell)} |u|_{W_q^\ell(\mathbb{R}^N)}, \tag{3.121}$$

where C depends on N, m, and k. The lemma follows by summing this inequality over all $|\alpha| = k$. $\square$

As an additional application of our results on mollification, we provide a negative norm inverse property for S_h. The inverse property assumed for S_h bounds a positive norm of a function in terms of a norm with lower order or lower exponent, or possibly both. Notice that we must start with a positive norm on the left side, but we can have negative norms on the right side.

The standard proofs of the inverse property for the Lagrange finite element spaces, such as those in [5, Theorem 4.5.11] and [32, Proposition 3.1], proceed by first proving the property on the unit simplex, where all norms are equivalent, then mapping this result to an element using scaling inequalities, and finally summing up the contributions over the elements in question. If we try to prove an inverse property with a negative norm on the left side, the scaling inequalities forbid us from proceeding in this manner. Evidently we need another approach.

We exhibit the following negative norm inverse property as a consequence of the standard inverse assumption on S_h of Equation 3.5, so it applies to more general finite element spaces than just the Lagrange spaces. It is explicitly stated in [31, Remark 4.1] that no inverse property with a negative norm on the left side is known for the Lagrange spaces, except in 1 dimension.

Lemma 3.8. *Suppose that U_1 and U_3 are open subsets of $\mathbb{R}^N$ with $U_1 \subset U_3$ and* $\operatorname{dist}(U_1, \partial U_3) \geq 2C_{sep}h$*. If $1 \leq q \leq p \leq \infty$, $k \in 1 : r$, $\ell \in k : r$, and $\chi \in S_h$, then*

$$\|\chi\|_{W_p^{-k}(U_1,\Omega)} \leq Ch^{-N(1/q-1/p)-(\ell-k)}\|\chi\|_{W_q^{-\ell}(U_3,\Omega)}, \tag{3.122}$$

where C depends on N, r, C_{ap}, and C_{sep}.

Proof. In this proof, let C denote different positive constants that depend on N, r, C_{ap}, and C_{sep}.

First observe that there exists an open subset U_2 of $\mathbb{R}^N$ such that $U_1 \subset U_2 \subset U_3$, $\operatorname{dist}(U_1, \partial U_2) \geq C_{\text{sep}}h$, and $\operatorname{dist}(U_2, \partial U_3) \geq C_{\text{sep}}h$. By the general definition of the negative norm,

$$\|\chi\|_{W_p^{-k}(U_1,\Omega)} = \sup_{\substack{\phi \in C_0^\infty(U_1) \\ \|\phi\|_{W_{p'}^k(U_1)}=1}} \left|\int_{U_1\cap\Omega} \chi\phi\right|. \tag{3.123}$$

Let $\phi \in C_0^\infty(U_1)$ have $\|\phi\|_{W_{p'}^k(U_1)} = 1$. Let m be an integer such that $2m + 2 \geq r$, and let d be such that $d\sqrt{N}(m + 1) = C_{\text{sep}}h$. By Theorem 3.3, there exists some $\psi \in C_0^\infty(U_2)$ such that

$$\|\phi - \psi\|_{L_{p'}(U_2)} \leq Cd^k\|\phi\|_{W_{p'}^k(U_1)} \tag{3.124}$$

and

$$\|\psi\|_{W_{q'}^\ell(U_2)} \leq Cd^{-N(1/p'-1/q')-(\ell-k)}\|\phi\|_{W_{p'}^k(U_1)}. \tag{3.125}$$

Observe that

$$\begin{aligned} \left|\int_{U_1\cap\Omega} \chi\phi\right| &= \left|\int_{U_2\cap\Omega} \chi\phi\right| \\ &\leq \left|\int_{U_2\cap\Omega} \chi(\phi - \psi)\right| + \left|\int_{U_2\cap\Omega} \chi\psi\right|. \end{aligned} \tag{3.126}$$

We estimate the first term using Hölder's inequality, the inverse assumption on S_h, and Equation 3.124,

$$
\begin{aligned}
|\int_{U_2\cap\Omega}\chi(\phi-\psi)| &\le \|\chi\|_{L_p(U_2\cap\Omega)}\|\phi-\psi\|_{L_{p'}(U_2\cap\Omega)} \\
&\le C\Big(h^{-\ell-N(1/q-1/p)}\|\chi\|_{W_q^{-\ell}(U_3\cap\Omega)}\Big)\Big(d^k\|\phi\|_{W_{p'}^k(U_1)}\Big) \\
&\le Ch^{-N(1/q-1/p)-(\ell-k)}\|\chi\|_{W_q^{-\ell}(U_3\cap\Omega)}.
\end{aligned}
\tag{3.127}
$$

To estimate the second term on the right side of Equation 3.126, we first recall that $\mathrm{supp}(\psi)\subset U_2$, and therefore, by the general definition of the negative norm,

$$
|\int_{U_2\cap\Omega}\chi\psi| \le \|\chi\|_{W_q^{-\ell}(U_2,\Omega)}\|\psi\|_{W_{q'}^{\ell}(U_2)}. \tag{3.128}
$$

By Equation 3.125,

$$
\begin{aligned}
\|\psi\|_{W_{q'}^{\ell}(U_2)} &\le Cd^{-N(1/p'-1/q')-(\ell-k)}\|\phi\|_{W_{p'}^k(U_1)} \\
&\le Ch^{-N(1/q-1/p)-(\ell-k)}.
\end{aligned}
\tag{3.129}
$$

Putting together Equations 3.128 and 3.129,

$$
|\int_{U_2\cap\Omega}\chi\psi| \le Ch^{-N(1/q-1/p)-(\ell-k)}\|\chi\|_{W_q^{-\ell}(U_2,\Omega)}. \tag{3.130}
$$

The lemma follows from Equations 3.123, 3.126, 3.127, and 3.130. □

The result is strengthened by the presence of the general negative norm on the left and weakened by the presence of the general negative norm on the right, with respect to how it would read if the usual negative norms were used.

Consider what would have happened if we did not have the theory of mollifiers. In order to obtain Equation 3.129, we could have turned to the inverse assumption on S_h and taken $\psi = I_h\phi$. There are two difficulties with this.

First, since $I_h\phi$ is only defined on Ω, we can not use the general negative norm in Equation 3.128. In fact, the only sensible thing we can do is to use the usual negative norms in both Equations 3.123 and 3.128.

That is, we assume that $\mathrm{supp}(\phi) \subset U_1 \cap \Omega$ in Equation 3.123. Equation 3.128 then requires that $\mathrm{supp}(I_h \phi) \subset U_2 \cap \Omega$. This will be true if I_h respects homogeneous boundary conditions, but we have made no such assumption here.

Second, functions in S_h are guaranteed only one order of differentiability, unlike mollifications of infinitely differentiable functions, which are themselves infinitely differentiable. In Equation 3.129, we would be restricted to having $\ell = k = 1$. It is for this exact same reason that the negative norm error estimate in [9, Lemma 5.4] only handles the W_∞^{-1} case and has no straightforward extension to the W_∞^{-k} cases for any integer $k > 1$.

3.6 Future Work

In Equation 3.22, where the positive and negative norm estimates are compared, it would be satisfying if the logarithmic factor ℓ_h in the positive norm cases could be reduced to $\ell_{h/H}$. This would result in an unbroken pattern with the negative norm cases. It is interesting to note that, as announced in [22, Section 2], it was initially thought that the negative norms would have the more damaging logarithmic factor than the positive norms.

We avoided stating the comparison in Equation 3.22 for $F \neq 0$ because the results do not form a very nice pattern in this case. Furthermore, although the second perturbation terms in the estimates of Equations 3.9 and 3.11 are readily compared with that of Equation 3.26, the same is far from true for the first perturbation terms. Along these same lines, it should be mentioned that the positive norm estimates for a local problem in [20, Theorems 1.1 and 1.2] take F in norms analogous to those in [19, Theorems 2.2 and 3.2]. However, in [21, Theorems 1 and 2], where a local problem is also considered, F is taken in sharper but substantially more complicated norms. These estimates have an obvious analogue for the global

Neumann problem. The norms of F appearing in Equations 3.9 and 3.11 were chosen because they are the simplest to state and make results easiest to prove. They are not necessarily the sharpest or the most natural, and many other options are available.

Unfortunately, a more careful analysis of the perturbation functional in the negative norm estimates is not merely a matter of obtaining sharper estimates for $v - I_h v$ and v than those of Equations 3.42 and 3.43. The handling of the perturbation functional in both of the positive norm estimates comes into play and complicates matters, as can be seen from Equations 3.45 and 3.47. Since there is not yet any application of a better handling of the perturbation functional, we question whether this is really worth investigating.

It would be nice to have an application for which Theorem 3.1 works but the weaker result of [9, Lemma 5.4] does not. At this point, the only application of either is to the pointwise a posteriori error estimators of Chapter 4, and, for this, both results suffice.

CHAPTER 4

ASYMPTOTICALLY EXACT L_∞ A POSTERIORI ERROR ESTIMATORS FOR THE FINITE ELEMENT METHOD

4.1 Introduction and Statement of Results

Let $N \geq 2$ be an integer and let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. For $i, j \in 1 : N$, let $a_{i,j}, b_i, c : \bar{\Omega} \to \mathbb{R}$ be sufficiently smooth. Define the bilinear form A on functions $v, w : \Omega \to \mathbb{R}$ by

$$A(v, w) = \int_\Omega \left(\sum_{i,j=1}^{N} a_{i,j} D_i v D_j w + \sum_{i=1}^{N} b_i D_i v w + c v w \right). \tag{4.1}$$

We assume that A is coercive over $W_2^1(\Omega)$. That is, there exists a constant $C_{\text{co}} > 0$ such that, if $v \in W_2^1(\Omega)$ then

$$A(v, v) \geq C_{\text{co}} \|v\|^2_{W_2^1(\Omega)}. \tag{4.2}$$

We also assume that A is uniformly elliptic on Ω. That is, there exists a constant $C_{\text{ell}} > 0$ such that, if $x \in \Omega$ and $\xi \in \mathbb{R}^N$ then

$$\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq C_{\text{ell}} |\xi|^2. \tag{4.3}$$

Let $h > 0$ be sufficiently small and let $\underline{c}, \bar{c} > 0$. Let T_h be a finite collection of subsets of Ω for which the following hold.

1. The union of the elements of T_h is $\bar{\Omega}$.

2. Elements of T_h are simplices whose faces are straight unless they meet $\partial\Omega$.

3. Elements of T_h meet face-to-face or not at all.

4. Each element of T_h contains a ball of radius $\underline{c}h$ and is contained in a ball of radius $\bar{c}h$.

Let $r \geq 3$ be an integer and let S_h denote the set of $\chi \in C^0(\Omega)$ such that, if $\tau \in T_h$ then $\chi \in \Pi^{r-1}(\tau)$.

Let $u \in W_\infty^{r+1}(\Omega)$, $u_h \in S_h$, define $e = u - u_h$, and assume that

$$A(e, \chi) = 0 \tag{4.4}$$

for all $\chi \in S_h$. Let $h \leq H \leq 1$ and let U be an open subset of Ω with $\operatorname{diam}(U) \leq H$. Let $0 < \epsilon < 1$ and define

$$m' = \frac{h}{H}\ell_{r=3,h,h/H} + \left(\frac{H}{h}\right)^{r+1} h^\epsilon + h^\epsilon \ell_{r=3,h,h/H}. \tag{4.5}$$

Assume that there exists an open subset V of $\mathbb{R}^N$ and a constant κ such that $U \subset V$ and $\operatorname{dist}(U, \partial V) \leq \kappa H$. Assume furthermore that there exists an operator P_H on $W_\infty^1(U)$ and a constant C_P such that, if $v \in W_\infty^{r+1}(\Omega)$ then

$$\|v - P_H v\|_{L_\infty(U)} \leq C_P H^{r+1} \|v\|_{W_\infty^{r+1}(\Omega)}, \tag{4.6}$$

and if $v \in W_\infty^1(V \cap \Omega)$ then

$$\|P_H v\|_{L_\infty(U)} \leq C_P H^{-1} \|v\|_{W_\infty^{-1}(V,\Omega)}. \tag{4.7}$$

Let $\tau \in T_h$ be such that $\tau \subset U$. Define

$$\mathcal{E}(\tau) = \|u_h - P_H u_h\|_{L_\infty(\tau)}. \tag{4.8}$$

We say that τ is nondegenerate if

$$|u|_{W_\infty^r(\tau)} \geq h^{1-\epsilon} \|u\|_{W_\infty^{r+1}(\Omega)} \tag{4.9}$$

and degenerate if

$$|u|_{W_\infty^r(\tau)} \leq h^{1-\epsilon} \|u\|_{W_\infty^{r+1}(\Omega)}. \tag{4.10}$$

The following theorem is our main result.

Theorem 4.1. *There exist constants $C, C' > 0$ such that the following hold with $m = C'm'$.*

1. *If τ is nondegenerate then*

$$C^{-1}h^r|u|_{W^r_\infty(\tau)} \leq \|e\|_{L_\infty(\tau)} \leq Ch^r|u|_{W^r_\infty(\tau)}, \tag{4.11}$$

$$\|e\|_{L_\infty(\tau)} \geq C^{-1}h^{r+1-\epsilon}\|u\|_{W^{r+1}_\infty(\Omega)}, \tag{4.12}$$

and

$$\frac{1}{1+m}\mathcal{E}(\tau) \leq \|e\|_{L_\infty(\tau)}, \tag{4.13}$$

and, if, in addition, $m < 1$, then

$$\|e\|_{L_\infty(\tau)} \leq \frac{1}{1-m}\mathcal{E}(\tau). \tag{4.14}$$

2. *If τ is degenerate then*

$$\|e\|_{L_\infty(\tau)} \leq Ch^{r+1-\epsilon}\|u\|_{W^{r+1}_\infty(\Omega)} \tag{4.15}$$

and

$$\mathcal{E}(\tau) \leq (C+m)h^{r+1-\epsilon}|u|_{W^{r+1}_\infty(\Omega)}. \tag{4.16}$$

3. *If*

$$\|e\|_{L_\infty(\tau)} \geq Ch^{r+1-\epsilon}\|u\|_{W^{r+1}_\infty(\Omega)} \tag{4.17}$$

then τ is nondegenerate.

The constants C and C' depend on N, Ω, various norms of the coefficients of A, C_{co}, C_{ell}, $\underline{c}$, $\overline{c}$, r, and κ. In addition, C' depends on C_P.

4.2 Motivation

First we motivate u, u_h, and their relationship in Equation 4.4.

Define the second-order differential operator L on functions $v : \Omega \to \mathbb{R}$ by

$$Lv = -\sum_{i,j=1}^{N} D_i(a_{i,j} D_j v) + \sum_{i=1}^{N} b_i D_i v + cv. \tag{4.18}$$

The corresponding co-normal derivative operator B is defined on functions $v : \bar{\Omega} \to \mathbb{R}$ by

$$Bv = \sum_{i,j=1}^{N} a_{i,j} (\nu_\Omega)_j D_i v. \tag{4.19}$$

We typically think of $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ as the solution of the classical homogeneous Neumann problem

$$\begin{aligned} Lu &= f \quad \text{on } \Omega \\ Bu &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.20}$$

where $f \in C^0(\Omega)$ is given.

We may also think of $u \in W_2^1(\Omega)$ as the solution of the weak problem

$$A(u, v) = \int_\Omega fv \tag{4.21}$$

for all $v \in W_2^1(\Omega)$, where $f \in L_2(\Omega)$ is given. By integration by parts, it is easily seen that, if u is a solution of the classical problem, then it is a solution of the weak problem. Notice that the weak problem admits solutions with less regularity than the classical problem.

In general, it is not feasible to find an explicit formula for u, so we resort to numerical methods to approximate it. The space S_h is the Lagrange finite element space, consisting of continuous functions on Ω which are polynomials of degree at most $r - 1$ on each element of the quasiuniform partition T_h. This space has the standard approximation, inverse, and superapproximation properties. The finite element approximation of the solution of Equation 4.21 is the unique solution $u_h \in S_h$ of the finite-dimensional linear system

$$A(u_h, \chi) = \int_\Omega f\chi \tag{4.22}$$

for all $\chi \in S_h$. From Equations 4.21 and 4.22, we obtain Equation 4.4.

In regions where the finite element error e is large, one would want to refine the partition in order to obtain a more accurate result, and in regions where e is small, refining the partition would be needlessly expensive. However, since u is unknown, it is not obvious where e is large and where it is small.

This motivates $\mathcal{E}$, which is a local L_∞ error estimator on elements in U. We say that $\mathcal{E}$ is asymptotically equivalent if there exists a constant $c > 0$ such that, if h is sufficiently small then

$$c^{-1}\mathcal{E}(\tau) \leq \|e\|_{L_\infty(\tau)} \leq c\mathcal{E}(\tau) \tag{4.23}$$

for all $\tau \in T_h$ with $\tau \subset U$. If this holds with $c \to 1$ as $h \to 0^+$, we say that $\mathcal{E}$ is asymptotically exact.

As defined in Equation 4.8, $\mathcal{E}$ is a posteriori in nature because it involves the approximate solution u_h, and thus can not be computed until after the finite element solution has been obtained. It is local in nature because it only takes into account the values of u_h on $V \cap \Omega$, and not on all of Ω.

Once u_h is known, $\mathcal{E}$ is only as difficult to compute as $P_H u_h$ is. We see from Equation 4.6 that P_H is an approximate identity operator and can approximate functions to higher order than the finite element space can. Equation 4.7 is an inverse or smoothing property. An example of an approximate identity operator with these properties is given in Section 4.6. If $m < 1$ then $P_H u_h$ approximates u better than u_h does on nondegenerate elements, as we will see in Equation 4.36. This is the underlying reason why our error estimator works.

From Equation 4.10, we see that, as $h \to 0^+$, degeneracy can only persist in regions in which all the rth order derivatives of u vanish. Therefore, for typical problems, we would expect degeneracy to be rare.

Theorem 4.1 has three parts. First, it gives some consequences of nondegeneracy. These can be used to prove asymptotic equivalence and exactness of $\mathcal{E}$. Second, it gives some consequences of degeneracy. Third, it gives a condition which implies nondegeneracy.

Equation 4.11 indicates that, in the nondegenerate case, the error behaves exactly as the interpolation error. That is, it is free of pollution.

We now give conditions under which $\mathcal{E}$ is asymptotically equivalent and asymptotically exact. Assume that, for sufficiently small h, every $\tau \in T_h$ with $\tau \subset U$ is nondegenerate. If m' stays bounded, Equation 4.13 gives the first inequality in Equation 4.23. If $m < 1$, Equation 4.14 gives the second inequality in Equation 4.23. If $0 \leq m_0 < 1$ and $m \leq m_0$ for all sufficiently small h then $\mathcal{E}$ is asymptotically equivalent. If $m \to 0$ as $h \to 0^+$ then $\mathcal{E}$ is asymptotically exact. We now give a simple example of a relationship between H and h which leads to asymptotic equivalence, and a more complicated example which leads to asymptotic exactness.

To obtain asymptotic equivalence for $r \geq 4$, consider taking $H = kh$ for k fixed. Then

$$m = C'\Big(\frac{1}{k}(1+\log k) + k^{r+1}h^\epsilon + h^\epsilon(1+\log k)\Big). \tag{4.24}$$

Let k be sufficiently large that $\frac{1}{k}(1+\log k) \leq \frac{1}{3C'}$. For all sufficiently small h, we have $h^\epsilon(k^{r+1}+1+\log k) \leq \frac{1}{3C'}$, and thus $m \leq 2/3$.

To obtain asymptotic exactness, consider taking $H = h^{1-k}$, where $k < \frac{\epsilon}{r+1}$ is fixed. Then

$$m = \begin{cases} C'\left(h^k\left(1+k\log\dfrac{1}{h}\right) + h^{-k(r+1)+\epsilon} + h^\epsilon\left(1+k\log\dfrac{1}{h}\right)\right), & \text{if } r \geq 4 \\ C'(h^k\ell_h + h^{-k(r+1)+\epsilon} + h^\epsilon\ell_h), & \text{if } r = 3. \end{cases} \tag{4.25}$$

The first and third terms obviously go to 0 as $h \to 0^+$. Since $\epsilon > k(r+1)$, the second term does the same. Therefore $m \to 0$ as $k \to 0^+$.

It may seem disappointing that we do not establish asymptotic equivalence in the presence of degenerate elements. However, the extreme degenerate case is when $e = 0$ on an entire element, and detection of this event appears to be difficult. Furthermore, the degenerate case, unlike the nondegenerate one, can occur when the error in one element is mostly due to pollution from elements far away. It appears to be difficult to construct an equivalent estimator in the case of pollution, especially one which is local in nature.

An element being degenerate does not typically mean that the error is large on the element. To the contrary, if m stays bounded then, by Equations 4.15 and 4.16, both the error and the estimator behave better than the interpolation error.

We also point out here that Equations 4.12 and 4.17 together show that non-degeneracy is equivalent to

$$\|e\|_{L_\infty(\tau)} \gtrsim h^{r+1-\epsilon} \|u\|_{W_\infty^{r+1}(\Omega)}. \tag{4.26}$$

4.3 Relationship to Prior Work

The approximation and inverse assumptions on an approximate identity operator in Equations 4.6 and 4.7 are analogous to those of [15, Equations 2.3 and 2.2] on an approximate gradient operator. The construction of an estimator for e from the approximate identity operator in Equation 4.8 is analogous to the construction of an estimator for De from the approximate gradient operator in [15, Equation 2.4]. The results of Theorem 4.1, which concern degeneracy and the estimator for e, are analogous to those of [15, Theorem 2.1 and Corollary 2.2], which concern degeneracy and the estimator for De.

The main technical tools for proving Theorem 4.1 are L_∞ and W_∞^{-1} error expansion inequalities. Weighted error estimates in L_∞ and W_∞^{-1} are given in [19, Theorem 2.1] and Theorem 3.1, respectively. These are then combined with Proposition 1.7, which gives expansion inequalities for weighted norms. Similarly, the proofs of [15, Theorem 2.1 and Corollary 2.2] require W_∞^1 and L_∞ error expansion inequalities. These are given in [19, Theorems 4.2 and 4.1], respectively. Although the L_∞ error expansion inequality of [15, Equation 3.3] mistakenly applies [19, Theorem 4.1] with a noninteger weight power, it may be obtained from [19, Theorem 3.1] and Proposition 1.7.

In Section 4.6 we give an example of an approximate identity operator and prove that it satisfies the required properties. This is not straightforward. In [15, Examples 1.1–1.3], three examples of approximate gradient operators are given. All three are quite easily shown to satisfy the properties of [15, Section 4].

4.4 Proof of Theorem

We first state a lemma which will imply Theorem 4.1. This is analogous to [15, Proposition 3.1], which implies [15, Theorem 2.1].

Lemma 4.2. *There exist constants* $C_1, C_2, C_3, C_4 > 0$ *such that*

$$\|u - P_H u_h\|_{L_\infty(\tau)} \leq C_1 C_P m' \Big(h^r |u|_{W^r_\infty(\tau)} + h^{r+1-\epsilon} \|u\|_{W^{r+1}_\infty(\Omega)} \Big), \tag{4.27}$$

$$\|e\|_{L_\infty(\tau)} \geq C_2 h^r |u|_{W^r_\infty(\tau)} - C_3 h^{r+1} \|u\|_{W^{r+1}_\infty(\Omega)}, \tag{4.28}$$

and

$$\|e\|_{L_\infty(\tau)} \leq C_4 \Big(h^r |u|_{W^r_\infty(\tau)} + h^{r+1-\epsilon} \|u\|_{W^{r+1}_\infty(\Omega)} \Big). \tag{4.29}$$

The constants C_1, C_2, C_3, *and* C_4 *depend on* N, Ω, *various norms of the coefficients of* A, C_{co}, C_{ell}, $\underline{c}$, $\overline{c}$, r, *and* κ.

Notice that Equation 4.28 contains two constants, whereas the analogous estimate in the first inequality of [15, Equation 3.2] contains only one constant. As we will explain, in the proof of Equation 4.28, the first inequality of [15, Lemma 3.3] is in error, and is responsible for this discrepancy.

We now see how Theorem 4.1 follows from Lemma 4.2. We defer the proof of this lemma to the next section.

First consider part 1. That is, we assume that Equation 4.9 holds. By Equations 4.28 and 4.9,

$$\begin{aligned} \|e\|_{L_\infty(\tau)} &\geq C_2 h^r |u|_{W^r_\infty(\tau)} - C_3 h^{r+1} \|u\|_{W^{r+1}_\infty(\Omega)} \\ &\geq (C_2 - C_3 h^\epsilon) h^r |u|_{W^r_\infty(\tau)}. \end{aligned} \tag{4.30}$$

For h sufficiently small, $C_3 h^\epsilon \leq C_2/2$, so

$$\|e\|_{L_\infty(\tau)} \geq \frac{1}{2} C_2 h^r |u|_{W^r_\infty(\tau)}. \tag{4.31}$$

If we take $C \geq 2/C_2$, we obtain the first inequality in Equation 4.11.

By Equations 4.29 and 4.9,

$$\begin{aligned} \|e\|_{L_\infty(\tau)} &\leq C_4 \left(h^r |u|_{W^r_\infty(\tau)} + h^{r+1-\epsilon} \|u\|_{W^{r+1}_\infty(\Omega)} \right) \\ &\leq 2 C_4 h^r |u|_{W^r_\infty(\tau)}. \end{aligned} \tag{4.32}$$

If we take $C \geq 2C_4$, we obtain the second inequality in Equation 4.11.

By the first inequality in Equation 4.11, along with Equation 4.9,

$$\begin{aligned} \|e\|_{L_\infty(\tau)} &\geq C^{-1} h^r |u|_{W^r_\infty(\tau)} \\ &\geq C^{-1} h^{r+1-\epsilon} \|u\|_{W^{r+1}_\infty(\Omega)}, \end{aligned} \tag{4.33}$$

which proves Equation 4.12.

By Equations 4.27 and 4.9,

$$\begin{aligned} \|u - P_H u_h\|_{L_\infty(\tau)} &\leq C_1 C_P m' \left(h^r |u|_{W^r_\infty(\tau)} + h^{r+1-\epsilon} \|u\|_{W^{r+1}_\infty(\Omega)} \right) \\ &\leq 2 C_1 C_P m' h^r |u|_{W^r_\infty(\tau)}. \end{aligned} \tag{4.34}$$

Combining this with Equation 4.31,

$$\|u - P_H u_h\|_{L_\infty(\tau)} \leq \frac{4 C_1 C_P m'}{C_2} \|e\|_{L_\infty(\tau)}. \tag{4.35}$$

If we take $C' \geq 4C_1C_P/C_2$, we obtain

$$\|u - P_H u_h\|_{L_\infty(\tau)} \leq m\|u - u_h\|_{L_\infty(\tau)}. \tag{4.36}$$

Therefore,

$$\begin{aligned} \|u_h - P_H u_h\|_{L_\infty(\tau)} &\leq \|u - u_h\|_{L_\infty(\tau)} + \|u - P_H u_h\|_{L_\infty(\tau)} \\ &\leq (1+m)\|u - u_h\|_{L_\infty(\tau)}, \end{aligned} \tag{4.37}$$

which proves Equation 4.13, and

$$\begin{aligned} \|u_h - P_H u_h\|_{L_\infty(\tau)} &\geq \|u - u_h\|_{L_\infty(\tau)} - \|u - P_H u_h\|_{L_\infty(\tau)} \\ &\geq (1-m)\|u - u_h\|_{L_\infty(\tau)}, \end{aligned} \tag{4.38}$$

which proves Equation 4.14.

Next consider part 2. That is, we assume that Equation 4.10 holds. By Equations 4.29 and 4.10,

$$\begin{aligned} \|e\|_{L_\infty(\tau)} &\leq C_4\Big(h^r|u|_{W_\infty^r(\tau)} + h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}\Big) \\ &\leq 2C_4 h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}. \end{aligned} \tag{4.39}$$

If we take $C \geq 2C_4$, we obtain Equation 4.15.

By Equations 4.27 and 4.10,

$$\begin{aligned} \|u - P_H u_h\|_{L_\infty(\tau)} &\leq C_1C_P m'\Big(h^r|u|_{W_\infty^r(\tau)} + h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}\Big) \\ &\leq 2C_1C_P m' h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}. \end{aligned} \tag{4.40}$$

If we take $C' \geq 2C_1C_P$, we obtain

$$\|u - P_H u_h\|_{L_\infty(\tau)} \leq m h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}. \tag{4.41}$$

Combining this with Equation 4.15,

$$\begin{aligned} \|u_h - P_H u_h\|_{L_\infty(\tau)} &\leq \|u - u_h\|_{L_\infty(\tau)} + \|u - P_H u_h\|_{L_\infty(\tau)} \\ &\leq (C+m)h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}, \end{aligned} \tag{4.42}$$

which proves Equation 4.16.

To obtain all the results of parts 1 and 2 together, we take $C = \max(2/C_2, 2C_4)$ and $C' = \max(4C_1C_P/C_2, 2C_1C_P)$.

Finally we consider part 3. That is, we assume that Equation 4.17 holds. By Equation 4.29,

$$\begin{aligned} Ch^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)} &\leq \|e\|_{L_\infty(\tau)} \\ &\leq C_4\left(h^r|u|_{W_\infty^r(\tau)} + h^{r+1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}\right). \end{aligned} \tag{4.43}$$

Dividing by Ch^r and using the fact that $C \geq 2C_4$, we see that

$$h^{1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)} \leq \frac{1}{2}\Big(|u|_{W_\infty^r(\tau)} + h^{1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}\Big). \tag{4.44}$$

Kicking back the last term on the right side, we obtain

$$\frac{1}{2}h^{1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)} \leq \frac{1}{2}|u|_{W_\infty^r(\tau)}, \tag{4.45}$$

which proves Equation 4.9.

4.5 Proof of Lemma

In this section, we prove the three inequalities of Lemma 4.2. We let C denote different positive constants that depend on N, Ω, various norms of the coefficients of A, C_{co}, C_{ell}, $\underline{c}$, $\overline{c}$, r, and κ.

First we show Equation 4.27. By Equations 4.6 and 4.7,

$$\begin{aligned} \|u - P_H u_h\|_{L_\infty(\tau)} &\leq \|u - P_H u\|_{L_\infty(\tau)} + \|P_H(u - u_h)\|_{L_\infty(\tau)} \\ &\leq C_P\Big(H^{r+1}\|u\|_{W_\infty^{r+1}(\Omega)} + H^{-1}\|u - u_h\|_{W_\infty^{-1}(V,\Omega)}\Big). \end{aligned} \tag{4.46}$$

By Theorem 3.1,

$$\|u - u_h\|_{W_\infty^{-1}(V,\Omega)} \leq Ch^2\ell_{r=3,h,h/H} \inf_{\chi\in S_h} \|u - \chi\|_{W_\infty^1(\Omega),V,h^{r-3}H^{4-r},1}. \tag{4.47}$$

Using the approximation property of the Lagrange finite element space, as in [19, Equation 4.4],

$$\inf_{\chi \in S_h} \|u - \chi\|_{W^1_\infty(\Omega),V,h^{r-3}H^{4-r},1} \leq Ch^{r-1}|u|_{W^r_\infty(\Omega),V,h^{r-3}H^{4-r},1}. \tag{4.48}$$

Since $h^{r-3}H^{4-r} \leq H$ and $w \mapsto \sigma_{V,w}(x)$ is increasing for all $x \in \Omega$,

$$|u|_{W^r_\infty(\Omega),V,h^{r-3}H^{4-r},1} \leq |u|_{W^r_\infty(\Omega),V,H,1}. \tag{4.49}$$

If $x \in \Omega$ then $\operatorname{dist}(x, \tau) \leq \operatorname{dist}(x, V) + (\kappa + 1)H$. Therefore, by Proposition 1.7,

$$|u|_{W^r_\infty(\Omega),V,H,1} \leq C\Big(|u|_{W^r_\infty(\tau)} + H|u|_{W^{r+1}_\infty(\Omega)}\Big). \tag{4.50}$$

Putting together Equations 4.46, 4.47, 4.48, 4.49, and 4.50,

$$\begin{aligned} \|u - P_H u_h\|_{L_\infty(\tau)} \leq CC_P\Big(&H^{-1}h^{r+1}\ell_{r=3,h,h/H}|u|_{W^r_\infty(\tau)} \\ &+ (H^{r+1} + h^{r+1}\ell_{r=3,h,h/H})\|u\|_{W^{r+1}_\infty(\Omega)}\Big). \end{aligned} \tag{4.51}$$

By Equation 4.5,

$$\begin{aligned} H^{-1}h^{r+1}\ell_{r=3,h,h/H} &= h^r \frac{h}{H}\ell_{r=3,h,h/H} \\ &\leq h^r m' \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} H^{r+1} + h^{r+1}\ell_{r=3,h,h/H} &= h^{r+1-\epsilon}\left(\left(\frac{H}{h}\right)^{r+1} h^\epsilon + h^\epsilon \ell_{r=3,h,h/H}\right) \\ &\leq h^{r+1-\epsilon}m'. \end{aligned} \tag{4.53}$$

Combining Equations 4.51, 4.52, and 4.53,

$$\|u - P_H u_h\|_{L_\infty(\tau)} \leq CC_P m'\Big(h^r|u|_{W^r_\infty(\tau)} + h^{r+1-\epsilon}\|u\|_{W^{r+1}_\infty(\Omega)}\Big), \tag{4.54}$$

which proves Equation 4.27.

Next we show Equation 4.28. There exists an invertible affine linear map $F_\tau : T^N \to \tau$ such that $\tau = F_\tau(T^N)$. Let $\hat{u} = u \circ F_\tau$.

If $|\hat{u}|_{W^r_\infty(T^N)} = 0$ then, by a scaling inequality, $|u|_{W^r_\infty(\tau)} = 0$, so Equation 4.28 is trivial.

From now on, we assume that $|\hat{u}|_{W^r_\infty(T^N)} > 0$. Then there exist $|\beta| = r$ and $\hat{x}_0 \in T^N$ such that

$$|D^\beta \hat{u}(\hat{x}_0)| = |\hat{u}|_{W^r_\infty(T^N)}. \tag{4.55}$$

By a scaling inequality and the definition of the Lagrange finite element space,

$$\begin{aligned} \|e\|_{L_\infty(\tau)} &\geq \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\tau)} \\ &= \inf_{\hat{\chi} \in \Pi^{r-1}(T^N)} \|\hat{u} - \hat{\chi}\|_{L_\infty(T^N)}. \end{aligned} \tag{4.56}$$

Suppose that $\hat{\chi} \in \Pi^{r-1}(T^N)$. Then

$$\|\hat{u} - \hat{\chi}\|_{L_\infty(T^N)} \geq \|T^r_{\hat{x}_0}\hat{u} - \hat{\chi}\|_{L_\infty(T^N)} - \|\hat{u} - T^r_{\hat{x}_0}\hat{u}\|_{L_\infty(T^N)}. \tag{4.57}$$

The second term is easily estimated by Taylor's theorem,

$$\|\hat{u} - T^r_{\hat{x}_0}\hat{u}\|_{L_\infty(T^N)} \leq C|\hat{u}|_{W^{r+1}_\infty(T^N)}. \tag{4.58}$$

Now we turn to estimating the first term. For $|\alpha| \leq r$, define $p_\alpha \in \Pi^r(T^N)$ by $p_\alpha(\hat{x}) = \hat{x}^\alpha$ and let V_α denote the vector subspace of $\Pi^r(T^N)$ such that $\Pi^r(T^N) = \mathrm{span}(p_\alpha) \oplus V_\alpha$. The $L_\infty(T^N)$ distance between p_α and V_α can be bounded below by a positive constant that depends only on N and r.

If $|\alpha| \leq r$ then $\hat{x} \mapsto (\hat{x} - \hat{x}_0)^\alpha$ is a linear combination of p_γ with $\gamma \leq \alpha$. Therefore, by definition of the Taylor polynomial, $T^r_{\hat{x}_0}\hat{u} - \frac{1}{\beta!}D^\beta\hat{u}(\hat{x}_0)p_\beta$ is a linear combination of p_γ with $\gamma \neq \beta$, and is thus in V_β. Since $\hat{\chi} \in \Pi^{r-1}(T^N)$, certainly $\hat{\chi} \in V_\beta$. Now we see that $T^r_{\hat{x}_0}\hat{u} - \hat{\chi} - \frac{1}{\beta!}D^\beta\hat{u}(\hat{x}_0)p_\beta \in V_\beta$. Since $D^\beta\hat{u}(\hat{x}_0) \neq 0$, there exists some $\hat{\eta} \in V_\beta$ such that

$$T^r_{\hat{x}_0}\hat{u} - \hat{\chi} = \frac{1}{\beta!}D^\beta\hat{u}(\hat{x}_0)(p_\beta - \hat{\eta}). \tag{4.59}$$

Therefore, by Equation 4.55,

$$
\begin{aligned}
\|T^r_{\hat{x}_0}\hat{u} - \hat{\chi}\|_{L_\infty(T^N)} &= \frac{1}{\beta!}|D^\beta \hat{u}(\hat{x}_0)|\|p_\beta - \hat{\eta}\|_{L_\infty(T^N)} \\
&\geq C|\hat{u}|_{W^r_\infty(T^N)}.
\end{aligned}
\tag{4.60}
$$

By Equations 4.57, 4.60 and 4.58,

$$
\|\hat{u} - \hat{\chi}\|_{L_\infty(T^N)} \geq C|\hat{u}|_{W^r_\infty(T^N)} - C|\hat{u}|_{W^{r+1}_\infty(T^N)}. \tag{4.61}
$$

It is not possible, in general, to combine the two constants here into one. The analogous combination of constants in the proof of [15, Lemma 3.3] appears to be in error. Using scaling inequalities and Equations 4.56 and 4.61, we obtain Equation 4.28.

Finally we show Equation 4.29. Let $x \in \tau$. By [19, Theorem 2.1],

$$
|(u - u_h)(x)| \leq Ch \inf_{\chi \in S_h} \|u - \chi\|_{W^1_\infty(\Omega),\{x\},h,1-\epsilon}. \tag{4.62}
$$

Using the approximation property of the Lagrange finite element space, as in [19, Equation 4.4],

$$
\inf_{\chi \in S_h} \|u - \chi\|_{W^1_\infty(\Omega),\{x\},h,1-\epsilon} \leq Ch^{r-1}|u|_{W^r_\infty(\Omega),\{x\},h,1-\epsilon}. \tag{4.63}
$$

If $y \in \Omega$ then $\operatorname{dist}(y, \tau) \leq |x - y| + \bar{c}h$. Therefore, by Proposition 1.7,

$$
|u|_{W^r_\infty(\Omega),\{x\},h,1-\epsilon} \leq C\Big(|u|_{W^r_\infty(\tau)} + h^{1-\epsilon}|u|_{W^{r+1}_\infty(\Omega)}\Big). \tag{4.64}
$$

Putting together Equations 4.62, 4.63, and 4.64,

$$
|(u - u_h)(x)| \leq Ch^r\Big(|u|_{W^r_\infty(\tau)} + h^{1-\epsilon}|u|_{W^{r+1}_\infty(\Omega)}\Big). \tag{4.65}
$$

Taking the supremum over all $x \in \tau$ gives Equation 4.29.

4.6 An Approximate Identity Operator

Assume that there exist constants $c_1, c_2, c_3 > 0$ and a subset W of Ω, which is star-shaped with respect to a point, such that U contains a ball of radius $c_1 H$ and is contained in W, $\text{meas}_{N-1}(\partial U) \le c_2 H^{N-1}$, and $\text{diam}(W) \le c_3 H$. In this section, C will denote different positive constants that depend only on c_1, c_2, c_3, r, and N.

For $v \in L_1(U)$, the Riesz representation theorem guarantees that there exists a unique $P_H v \in \Pi^r(U)$ such that, for all $\chi \in \Pi^r(U)$,

$$\int_U (P_H v)\chi = \int_U v\chi. \tag{4.66}$$

That is, $P_H : L_1(U) \to \Pi^r(U)$ is the projection onto the space of polynomials of degree at most r on U. We will verify that Equations 4.6 and 4.7 hold with P_H defined by Equation 4.66, $\kappa = 0$, $V = U$, and C_P depending only on c_1, c_2, c_3, r, and N.

First we prove an inverse property. Since all norms are equivalent on the finite-dimensional vector space $\Pi^r(\mathbb{R}^N)$, we know that, for $k \in 0:1$, $\ell \in -1:0$, $1 \le p, q \le \infty$, and $\hat{\chi} \in \Pi^r(\mathbb{R}^N)$,

$$\|\hat{\chi}\|_{W_p^k(B_1(0))} \le C\|\hat{\chi}\|_{W_q^\ell(B_{c_1}(0))}. \tag{4.67}$$

Let x_0 be such that $B_{c_1 H}(x_0) \subset U$. Since $\text{diam}(U) \le H$, we must have $U \subset B_H(x_0)$. Define $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi(x) = (x - x_0)/H$. If $k \in 0:1$, $\ell \in -1:0$, $1 \le p, q \le \infty$, and $\chi \in \Pi^r(\mathbb{R}^N)$ then $\chi \circ \Phi^{-1} \in \Pi^r(\mathbb{R}^N)$, so, using scaling inequalities and Equation 4.67,

$$\begin{aligned}
\|\chi\|_{W_p^k(U)} &\le \|\chi\|_{W_p^k(B_H(x_0))} \\
&\le CH^{-k+N/p}\|\chi \circ \Phi^{-1}\|_{W_p^k(B_1(0))} \\
&\le CH^{-k+N/p}\|\chi \circ \Phi^{-1}\|_{W_q^\ell(B_{c_1}(0))} \\
&\le CH^{-(k-\ell)-N(1/p-1/q)}\|\chi\|_{W_q^\ell(B_{c_1 H}(x_0))} \\
&\le CH^{-(k-\ell)-N(1/p-1/q)}\|\chi\|_{W_q^\ell(U)}.
\end{aligned} \tag{4.68}$$

Next we show that P_H is bounded on $L_2(U)$. If $v \in L_2(U)$ then, by Equation 4.66,

$$\begin{aligned} \|P_H v\|^2_{L_2(U)} &= \int_U (P_H v)(P_H v) \\ &= \int_U v(P_H v) \\ &\leq \|v\|_{L_2(U)} \|P_H v\|_{L_2(U)}. \end{aligned} \tag{4.69}$$

If $P_H v \neq 0$ then, dividing by $\|P_H v\|_{L_2(U)}$, we obtain $\|P_H v\|_{L_2(U)} \leq \|v\|_{L_2(U)}$. If $P_H v = 0$ then this is trivial.

Next we show that P_H is bounded on $L_\infty(U)$. If $v \in L_\infty(U)$ then, using Equation 4.68, the fact that P_H is bounded on $L_2(U)$, and the measure inequality,

$$\begin{aligned} \|P_H v\|_{L_\infty(U)} &\leq CH^{-N/2} \|P_H v\|_{L_2(U)} \\ &\leq CH^{-N/2} \|v\|_{L_2(U)} \\ &\leq C\|v\|_{L_\infty(U)}. \end{aligned} \tag{4.70}$$

Now we show that P_H satisfies Equation 4.6. Let x be a point with respect to which W is star-shaped. Suppose that $v \in W^{r+1}_\infty(\Omega)$. Since $T^r_x v \in \Pi^r(U)$, $P_H T^r_x v = T^r_x v$. Therefore,

$$v - P_H v = (v - T^r_x v) - P_H(v - T^r_x v). \tag{4.71}$$

By Taylor's theorem,

$$\|v - T^r_x v\|_{L_\infty(W)} \leq CH^{r+1} |v|_{W^{r+1}_\infty(W)}. \tag{4.72}$$

Using Equation 4.71, the fact that P_H is bounded on $L_\infty(U)$, and Equation 4.72, we find that

$$\begin{aligned} \|v - P_H v\|_{L_\infty(U)} &\leq \|v - T^r_x v\|_{L_\infty(U)} + \|P_H(v - T^r_x v)\|_{L_\infty(U)} \\ &\leq C\|v - T^r_x v\|_{L_\infty(U)} \\ &\leq CH^{r+1} |v|_{W^{r+1}_\infty(W)}. \end{aligned} \tag{4.73}$$

This is slightly better than Equation 4.6 because the right side has the $(r+1)$st-order seminorm of v on W instead of the $(r+1)$st-order norm of v on Ω.

Next we show that P_H is bounded on $L_1(U)$. Suppose that $v \in L_1(U)$. If $\phi \in L_\infty(U)$ then, using Equation 4.66 twice, along with the fact that P_H is bounded on $L_\infty(U)$,

$$\begin{aligned}\int_U (P_H v)\phi &= \int_U v\phi \\ &= \int_U v(P_H\phi) \\ &\le \|v\|_{L_1(U)}\|P_H\phi\|_{L_\infty(U)} \\ &\le C\|v\|_{L_1(U)}\|\phi\|_{L_\infty(U)}.\end{aligned} \tag{4.74}$$

Therefore, by the extremal version of Hölder's inequality,

$$\begin{aligned}\|P_H v\|_{L_1(U)} &= \sup_{\substack{\phi\in L_\infty(U)\\ \|\phi\|_{L_\infty(U)}=1}} \int_U (P_H v)\phi \\ &\le C\|v\|_{L_1(U)}.\end{aligned} \tag{4.75}$$

Next we show that P_H is bounded on $W_1^1(U)$. Suppose that $v \in W_1^1(U)$ and let c denote the average value of v on U. Since $c \in \Pi^r(U)$, $P_H c = c$. Therefore, by Equation 4.68, the fact that P_H is bounded on $L_1(U)$, and Poincaré's inequality,

$$\begin{aligned}|P_H v|_{W_1^1(U)} &= |P_H(v-c)|_{W_1^1(U)} \\ &\le CH^{-1}\|P_H(v-c)\|_{L_1(U)} \\ &\le CH^{-1}\|v-c\|_{L_1(U)} \\ &\le C|v|_{W_1^1(U)}.\end{aligned} \tag{4.76}$$

Combined with Equation 4.75, this shows that P_H is bounded on $W_1^1(U)$.

Finally we show that P_H satisfies Equation 4.7. Suppose that $v \in L_\infty(U)$. We don't actually need to assume $v \in W_\infty^1(U)$ here. By Equation 4.68,

$$\|P_H v\|_{L_\infty(U)} \le CH^{-1}\|P_H v\|_{W_\infty^{-1}(U)}. \tag{4.77}$$

By definition of the negative norm,

$$\|P_H v\|_{W_\infty^{-1}(U)} = \sup_{\substack{\phi \in C_0^\infty(U) \\ \|\phi\|_{W_1^1(U)}=1}} |\int_U (P_H v)\phi|. \tag{4.78}$$

Let $\phi \in C_0^\infty(U)$ have $\|\phi\|_{W_1^1(U)} = 1$. Let $\epsilon < c_1 H/2$ and define $U_\epsilon = \{x \in U : \mathrm{dist}(x, \partial U) > \epsilon\}$. Let $\omega_\epsilon \in C_0^\infty(U)$ be such that $\omega_\epsilon = 1$ on U_ϵ and, for $i \in 0:1$,

$$|\omega_\epsilon|_{W_\infty^i(U)} \leq C\epsilon^{-i}. \tag{4.79}$$

Having made the assumption that $\mathrm{meas}_{N-1}(\partial U) \leq c_2 H^{N-1}$, we can conclude that $\mathrm{meas}_N(U \setminus U_\epsilon) \leq CH^{N-1}\epsilon$. Writing $1 = (1-\omega_\epsilon) + \omega_\epsilon$ has two advantages. First, $1-\omega_\epsilon$ and the first derivatives of ω_ϵ are only nonzero on the set $U \setminus U_\epsilon$, whose measure vanishes with ϵ. Second, the support of ω_ϵ is contained in U, which helps set up an estimate in a negative norm.

Using Equation 4.66 twice,

$$\begin{aligned} \int_U (P_H v)\phi &= \int_U v\phi \\ &= \int_U v(P_H \phi) \\ &= \int_U v(P_H\phi)(1-\omega_\epsilon) + \int_U v(P_H\phi)\omega_\epsilon. \end{aligned} \tag{4.80}$$

First we estimate the first term on the right side,

$$|\int_U v(P_H\phi)(1-\omega_\epsilon)| \leq \|v\|_{L_\infty(U)} \|P_H\phi\|_{L_\infty(U)} \|1-\omega_\epsilon\|_{L_1(U\setminus U_\epsilon)}. \tag{4.81}$$

By Equation 4.68 and the fact that P_H is bounded on $L_1(U)$,

$$\begin{aligned} \|P_H\phi\|_{L_\infty(U)} &\leq CH^{-N}\|P_H\phi\|_{L_1(U)} \\ &\leq CH^{-N}\|\phi\|_{L_1(U)} \\ &\leq CH^{-N}. \end{aligned} \tag{4.82}$$

By the measure inequality and Equation 4.79,

$$\|1-\omega_\epsilon\|_{L_1(U\setminus U_\epsilon)} \leq CH^{N-1}\epsilon. \tag{4.83}$$

By Equations 4.81, 4.82, and 4.83,

$$|\int_U v(P_H\phi)(1-\omega_\epsilon)| \le CH^{-1}\epsilon\|v\|_{L_\infty(U)}. \tag{4.84}$$

Now we estimate the second term on the right side of Equation 4.80. Since $\mathrm{supp}(\omega_\epsilon) \subset U$, we have by definition of the negative norm that

$$|\int_U v(P_H\phi)\omega_\epsilon| \le \|v\|_{W_\infty^{-1}(U)}\|(P_H\phi)\omega_\epsilon\|_{W_1^1(U)}. \tag{4.85}$$

The trick is to estimate

$$\begin{aligned}\|(P_H\phi)\omega_\epsilon\|_{W_1^1(U)} &\le \|P_H\phi\|_{W_1^1(U)}\|\omega_\epsilon\|_{L_\infty(U)} \\ &\quad + \|P_H\phi\|_{L_\infty(U)}\|\omega_\epsilon\|_{W_1^1(U)}.\end{aligned} \tag{4.86}$$

Using the fact that P_H is bounded on $W_1^1(U)$, along with Equation 4.79,

$$\begin{aligned}\|P_H\phi\|_{W_1^1(U)}\|\omega_\epsilon\|_{L_\infty(U)} &\le C\|\phi\|_{W_1^1(U)} \\ &\le C.\end{aligned} \tag{4.87}$$

Using Equation 4.68, the fact that P_H is bounded on $L_1(U)$, and Poincaré's inequality,

$$\begin{aligned}\|P_H\phi\|_{L_\infty(U)} &\le CH^{-N}\|P_H\phi\|_{L_1(U)} \\ &\le CH^{-N}\|\phi\|_{L_1(U)} \\ &\le CH^{1-N}|\phi|_{W_1^1(U)} \\ &\le CH^{1-N}.\end{aligned} \tag{4.88}$$

Using the fact that the derivatives of ω_ϵ are zero except on $U \smallsetminus U_\epsilon$, along with the measure inequality and Equation 4.79,

$$\begin{aligned}\|\omega_\epsilon\|_{W_1^1(U)} &\le \|\omega_\epsilon\|_{L_1(U)} + |\omega_\epsilon|_{W_1^1(U\smallsetminus U_\epsilon)} \\ &\le C\Big(H^N\|\omega_\epsilon\|_{L_\infty(U)} + H^{N-1}\epsilon|\omega_\epsilon|_{W_\infty^1(U)}\Big) \\ &\le C(H^N + H^{N-1}) \\ &\le CH^{N-1}.\end{aligned} \tag{4.89}$$

By Equations 4.85, 4.86, 4.87, 4.88, and 4.89,

$$|\int_U v(P_H\phi)\omega_\epsilon| \leq C\|v\|_{W_\infty^{-1}(U)}. \tag{4.90}$$

Putting together Equations 4.80, 4.84 and 4.90,

$$|\int_U (P_H v)\phi| \leq C\Big(H^{-1}\epsilon\|v\|_{L_\infty(U)} + \|v\|_{W_\infty^{-1}(U)}\Big). \tag{4.91}$$

Taking $\epsilon \to 0^+$, we are left with

$$|\int_U (P_H v)\phi| \leq C\|v\|_{W_\infty^{-1}(U)}. \tag{4.92}$$

Therefore, by Equation 4.78,

$$\|P_H v\|_{W_\infty^{-1}(U)} \leq C\|v\|_{W_\infty^{-1}(U)}. \tag{4.93}$$

This is equivalent to Equation 4.7 because $U \subset \Omega$.

4.7 Future Work

We have only demonstrated one approximate identity operator, in Section 4.6. Others may be considered. One idea would be to approximate the P_H defined in Equation 4.66 by numerical integration.

It would also be nice to have some numerical examples to demonstrate that the theory is actually useful in practice. The effectivity of the estimator, which is the ratio of the predicted error to the true error, is a standard measure of the quality of an estimator. An effectivity of 1 means that the estimator is perfect. Effectivities close to 0 or very large mean that the estimator is poor.

First, we would consider a smooth problem whose solution has non-negligible derivatives of order r. In this case, all elements would be nondegenerate, so we would expect our estimator to be accurate. That is, the effectivity would be close to 1.

After this easy example, we push the estimator to its theoretical limits. Next we would have a problem which has regions where the rth-order derivatives of the solution happen to be very small. Elements in these regions would be degenerate. We would expect that our estimator would be small and that the true error would be small, although these would not necessarily be commensurate.

Finally we would investigate a nonsmooth problem to observe the effects of pollution from outside our region of interest. Although our theory does not extend to this case, it is possible that we could still have decent results.

In all of these examples, we could vary the mesh size h and the patch size H and observe how this affects the results.

CHAPTER 5

WEIGHTED L_∞ AND W^1_∞ ERROR ESTIMATES FOR THE FINITE ELEMENT METHOD WITH SUPERPARAMETRIC ELEMENTS AND NUMERICAL INTEGRATION

5.1 Introduction and Statement of Results

Let $N \geq 2$ be an integer and let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary. For $i, j \in 1 : N$, let $a_{i,j}, b_i, c, f : \bar{\Omega} \to \mathbb{R}$ be sufficiently smooth. Define the bilinear form A on functions $v, w : \Omega \to \mathbb{R}$ by

$$A(v, w) = \int_\Omega \left(\sum_{i,j=1}^{N} a_{i,j} D_i v D_j w + \sum_{i=1}^{N} b_i D_i v w + c v w \right). \tag{5.1}$$

We assume that A is coercive over $\{v \in W^1_2(\Omega) : v = 0 \text{ on } \partial\Omega\}$. That is, there exists a constant $C_{\mathrm{co}} > 0$ such that, if $v \in W^1_2(\Omega)$ and $v = 0$ on $\partial\Omega$ then

$$A(v, v) \geq C_{\mathrm{co}} \|v\|^2_{W^1_2(\Omega)}. \tag{5.2}$$

We also assume that A is uniformly elliptic on Ω. That is, there exists a constant $C_{\mathrm{ell}} > 0$ such that, if $x \in \Omega$ and $\xi \in \mathbb{R}^N$ then

$$\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq C_{\mathrm{ell}} |\xi|^2. \tag{5.3}$$

Define the linear functional λ on functions $v : \Omega \to \mathbb{R}$ by

$$\lambda(v) = \int_\Omega f v. \tag{5.4}$$

Let $u \in W^1_2(\Omega)$ have $u = 0$ on $\partial\Omega$ and satisfy

$$A(u, v) = \lambda(v) \tag{5.5}$$

for all $v \in W^1_2(\Omega)$ with $v = 0$ on $\partial\Omega$.

Let $h > 0$ be sufficiently small, let $c > 0$, and let Ω_h be an open subset of $\mathbb{R}^N$. Let T_h be a finite collection of subsets of Ω_h for which the following hold.

1. The union of the elements of T_h is $\bar{\Omega}_h$.

2. If $\tau \in T_h$ then there exists an invertible map $F_\tau : T^N \to \tau$ such that $\tau = F_\tau(T^N)$.

3. Elements of T_h meet face-to-face or not at all.

4. If $\tau \in T_h$ then F_τ is sufficiently smooth on T^N, F_τ^{-1} is sufficiently smooth on τ, and $|F_\tau|_{(W^i_\infty(T^N))^N} \leq ch^i$ and $|F_\tau^{-1}|_{(W^i_\infty(\tau))^N} \leq ch^{-i}$ for all $i \in 0 : k$, where k is a sufficiently large integer.

Let $r \geq 2$ be an integer and let S_h denote the set of $\chi \in C^0(\Omega_h)$ such that, if $\tau \in T_h$ then $\chi \circ F_\tau \in \Pi^{r-1}(T^N)$.

Let $m \geq 2$ be an integer and assume that there exists a homeomorphism $\Phi_h : \Omega_h \to \Omega$ such that, for $i \in 0 : 1$,

$$|\Phi_h - I|_{(W^i_\infty(\Omega_h))^N} \leq ch^{m-i} \tag{5.6}$$

and

$$|\Phi_h^{-1} - I|_{(W^i_\infty(\Omega))^N} \leq ch^{m-i}. \tag{5.7}$$

Furthermore, assume that, if $\tau \in T_h$, then Φ_h is sufficiently smooth on τ, Φ_h^{-1} is sufficiently smooth on $\Phi_h(\tau)$, and $\|\Phi_h\|_{(W^k_\infty(\tau))^N} \leq c$ and $\|\Phi_h^{-1}\|_{(W^k_\infty(\Phi_h(\tau)))^N} \leq c$ for some sufficiently large integer k.

Let $q \geq 0$ be an integer and let $\hat{Q} \in (C^0(T^N))'$ be a quadrature rule of order q on T^N. That is, if $\hat{\chi} \in \Pi^q(T^N)$ then

$$\hat{Q}\hat{\chi} = \int_{T^N} \hat{\chi}. \tag{5.8}$$

Assume that $\hat{Q} \in (L_\infty(T^N))'$. For $\tau \in T_h$, define $Q_\tau \in (C^0(\tau))'$ by

$$Q_\tau v = \hat{Q}\Big((v \circ F_\tau) \det DF_\tau\Big). \tag{5.9}$$

For $i, j \in 1 : N$, let $\bar{a}_{i,j}, \bar{b}_i, \bar{c}, \bar{f} : \bar{\Omega}_h \to \mathbb{R}^N$ be sufficiently smooth extensions of $a_{i,j}$, b_i, c, and f, respectively. Define the bilinear form A_h on functions $\chi, \eta : \Omega_h \to \mathbb{R}$ by

$$A_h(\chi, \eta) = \sum_{\tau \in T_h} Q_\tau \left(\sum_{i,j=1}^{N} \bar{a}_{i,j} D_i \chi D_j \eta + \sum_{i=1}^{N} \bar{b}_i D_i \chi \eta + \bar{c} \chi \eta \right) \tag{5.10}$$

and define the linear functional λ_h on functions $\chi : \Omega_h \to \mathbb{R}$ by

$$\lambda_h(\chi) = \sum_{\tau \in T_h} Q_\tau(\bar{f}\chi). \tag{5.11}$$

Let $u_h \in S_h$ have $u_h = 0$ on $\partial\Omega_h$ and satisfy

$$A_h(u_h, \chi) = \lambda_h(\chi) \tag{5.12}$$

for all $\chi \in S_h$ with $\chi = 0$ on $\partial\Omega_h$.

We will let C denote different positive constants that depend on N, Ω, various norms of the coefficients of A, C_{co}, C_{ell}, c, r, m, q, and $\|\hat{Q}\|_{(L_\infty(T^N))'}$, in addition to other explicitly stated quantities.

The following two theorems are our main results.

Theorem 5.1. *If $x \in \Omega$, $0 \le s \le r-2$, $m = r+s$, and $q = \max\{2(r-2), 1\} + s$ then*

$$\|u - u_h \circ \Phi_h^{-1}\|_{L_\infty(\Omega),\{x\},h,s} \le Ch^r \ell_h \Big(\|u\|_{W_\infty^r(\Omega),\{x\},h,s} + h^s \|f\|_{W_\infty^{r+s}(\Omega)} \Big), \tag{5.13}$$

where C depends on s.

Theorem 5.2. *If $x \in \Omega$, $0 \le s \le r-1$, $m = r+s$, and $q = 2(r-2)+s$ then*

$$\|u - u_h \circ \Phi_h^{-1}\|_{W_\infty^1(\Omega),\{x\},h,s} \le Ch^{r-1} \ell_h \Big(\|u\|_{W_\infty^r(\Omega),\{x\},h,s} + h^s \|f\|_{W_\infty^{r-1+s}(\Omega)} \Big), \tag{5.14}$$

where C depends on s.

5.2 Motivation

In this section, we motivate u and u_h.

Define the differential operator L on functions $v : \Omega \to \mathbb{R}$ by

$$Lv = -\sum_{i,j=1}^{N} D_i(a_{i,j} D_j v) + \sum_{i=1}^{N} b_i D_i v + cv. \tag{5.15}$$

We typically think of $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ as the solution of the classical homogeneous Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.16}$$

where $f \in C^0(\Omega)$ is given. By integration by parts, it is easily seen that, if u is a solution of the classical problem, then it is a solution of the weak problem of Equation 5.5. Notice that the weak problem admits solutions with less regularity than the classical problem.

In general, it is not feasible to find an explicit formula for u, so we resort to numerical methods to approximate it. It is typically unrealistic to partition the domain Ω, so we settle for partitioning an approximation Ω_h of it. The space S_h consists of continuous functions on Ω_h which, when mapped from any element $\tau \in T_h$ back to the unit simplex T^N through F_τ, are polynomials of degree at most $r-1$. This space has the standard approximation, inverse, and superapproximation properties. It also respects homogeneous Dirichlet boundary conditions on $\partial\Omega_h$.

For $m = 2$, we could take each element of T_h to be a straight simplex that contains a ball of radius $\underline{c}h$ and is contained in a ball of radius $\overline{c}h$, where $\underline{c}, \overline{c} > 0$ are constants. In this case, we have $F_\tau \in (\Pi^1(T^N))^N$ for each $\tau \in T_h$. In [18, Section 3], it is shown how to modify such a partition in order to obtain one that works for general m, in which $F_\tau \in (\Pi^{m-1}(T^N))^N$ for each $\tau \in T_h$. In this case, each element of T_h is a curved simplex. The use of curved simplices allows Ω_h to be a better approximation of Ω without decreasing the size of the elements. Actually, only the faces of simplices which form part of $\partial\Omega_h$ need to be curved. For this partition, Equations 5.6 and 5.7 are obtained in [18, Section 5].

We say that Ω_h approximates the geometry of Ω to order m and S_h approximates functions to order r. The method is said to be subparametric or hypoparametric if $m < r$, superparametric or hyperparametric if $m > r$, and isoparametric if $m = r$.

Define the bilinear form A'_h on functions $\chi, \eta : \Omega_h \to \mathbb{R}$ by

$$A'_h(\chi, \eta) = \int_{\Omega_h} \left(\sum_{i,j=1}^{N} \bar{a}_{i,j} D_i\chi D_j\eta + \sum_{i=1}^{N} \bar{b}_i D_i\chi\eta + \bar{c}\chi\eta \right) \tag{5.17}$$

and define the linear functional λ'_h on functions $\chi : \Omega_h \to \mathbb{R}$ by

$$\lambda'_h(\chi) = \int_{\Omega_h} \bar{f}\chi. \tag{5.18}$$

The theoretical finite element approximation of the solution of Equation 5.5 is the solution $u'_h \in S_h$ with $u'_h = 0$ on $\partial\Omega$ of the finite-dimensional linear system

$$A'_h(u'_h, \chi) = \lambda'_h(\chi) \tag{5.19}$$

for all $\chi \in S_h$ with $\chi = 0$ on $\partial\Omega$. Notice that the boundary condition for the true solution u is imposed on $\partial\Omega$, but the boundary condition for the finite element solution u'_h is imposed on $\partial\Omega_h$.

In practice, the integrals involved in computing A'_h and λ'_h can not be done exactly. Instead, the typical procedure is as follows. First, an integral over Ω_h is written as the sum of integrals over all the elements. Second, integrals over each element are transferred to the unit simplex by the change of variables formula. Lastly, integrals over the unit simplex are approximated by the quadrature rule $\hat{Q}$.

For $v \in C^0(\tau)$, this means that we approximate

$$\begin{aligned}\int_{\Omega_h} v &= \sum_{\tau \in T_h} \int_\tau v \\ &= \sum_{\tau \in T_h} \int_{T^N} (v \circ F_\tau) \det DF_\tau \\ &\approx \sum_{\tau \in T_h} \hat{Q}\Big((v \circ F_\tau) \det DF_\tau\Big) \\ &= \sum_{\tau \in T_h} Q_\tau v.\end{aligned} \tag{5.20}$$

Approximating the integrals in Equations 5.17 and 5.18 in this manner, we see from Equations 5.10 and 5.11 that $A'_h(\chi, \eta) \approx A_h(\chi, \eta)$ and $\lambda'_h(\chi) \approx \lambda_h(\chi)$. Equation 5.12 arises by equating the approximations of the left and right sides of Equation 5.19.

When $\Omega_h = \Omega$, the finite element error has a clear meaning. We simply compare u to u_h on Ω. It is not true in general that $\Omega_h = \Omega$, so u will not be defined on all of Ω_h and u_h will not be defined on all of Ω. However, we could compare u to u_h on $\Omega \cap \Omega_h$. We could also extend u by zero to Ω_h and then compare u to u_h on all of Ω_h, or we could extend u_h by zero to Ω and then compare u to u_h on all of Ω.

Here we take a different approach, following [18, Section 4]. We compare u to $u_h \circ \Phi_h^{-1}$ on all of Ω. This means that u at a point $x \in \Omega$ is compared to u_h at the corresponding point $\Phi_h^{-1}(x) \in \Omega_h$. By Equation 5.7, the points x and $\Phi_h^{-1}(x)$ are at a distance up to Ch^m apart. Theorems 5.1 and 5.2 give estimates for global weighted L_∞ and W^1_∞ norms of $u - u_h \circ \Phi_h^{-1}$.

The condition $m = r + s$ means that we need to use superparametric elements if we want weighted estimates. The order of approximation of the geometry must exceed the order of approximation of functions by the weight power desired. Isoparametric elements are sufficient to obtain nonweighted estimates.

The condition $q = 2(r-2)+s$ means that we need to use an integration scheme of order $2(r-2)$ to obtain nonweighted estimates. The integration scheme must be one order higher than this for each weight power we desire. The exception to this is in the L_∞ estimate when $r = 2$. Here, no weight is possible. Instead of requiring an integration scheme of order 0, as the general pattern would predict, it appears that the scheme must be of order 1. Notice that the quadrature accuracy requirement does not involve m.

5.3 Relationship to Prior Work

Estimates for $u - u_h$ in $W_2^1(\Omega_h)$ and $L_2(\Omega_h)$ with the combined effect of using isoparametric elements and numerical integration are given in [8, Examples 6 and 7]. Here, u is extended by zero to Ω_h and only the $r = m \in 3 : 4$ cases are considered.

The combined effect of using isoparametric elements and numerical integration is again considered in [30, Theorem 1.1]. This time, u_h is extended by zero to Ω and $u - u_h$ is estimated in $L_\infty(\Omega)$. Only the specific case $N = 2$, $r = m = 3$ of isoparametric quadratic elements in the plane is examined. The quadrature rule integrates quadratics exactly, and thus has $q = 2$. The error is bounded in terms of $\|f\|_{W_1^3(\Omega)}$. In the present work, the error in this case is bounded in terms of $\|f\|_{W_\infty^3(\Omega)}$ and another term. A more careful analysis would give a sharper result, but this would complicate matters and obscure the main point of this work.

An estimate for $u - u_h$ in $W_2^1(\Omega)$ in the presence of quadratic isoparametric elements with second-order numerical integration is given in [7, Theorem 43.1]. That is, $r = m = 3$ and $q = 2$. The error is bounded in terms of $\|u\|_{W_2^3(\Omega)}$ and $\|f\|_{W_p^2(\Omega)}$ for some appropriate $p \geq 2$. In [7, Section 39], it is stated that this result could possibly be a fluke. At the time, it was expected that, in general, the use of curved elements would require more accurate quadrature schemes. In the present work, we show that this is not the case.

In [24, Theorem 5.1], estimates for $u - u_h$ in $L_\infty(\Omega \cap \Omega_h)$ are obtained using exact integration and straight elements. The results are consistent with those in the present work.

In [18, Section 5], optimal $W_2^1(\Omega)$ estimates for $u - u_h \circ \Phi_h^{-1}$ are obtained using isoparametric elements and exact integration.

It was found in [25, Section 5] that a quadrature rule of order $2(r-2)$ is required to preserve the interior W_∞^1 estimates for $u - u_h$ of [25, Theorem 1.2]. This result corresponds to the result of the present work in the case of no weight.

Similarly, it was found in [14, Theorem 1.4] that a quadrature rule of order $2(r-2)+s$ is required to preserve the interior weighted W_∞^1 estimates for $u - u_h$ of [20, Theorem 2.1], where s is the desired weight power. This result is consistent with the result of [25, Section 5] in the case of no weight, and corresponds to the general result of the present work.

It should be pointed out here that the L_∞ estimates are much trickier than the W_∞^1 estimates in several respects.

5.4 Proof of Results

Define the bilinear form A_h'' on functions $\chi, \eta : \Omega_h \to \mathbb{R}$ by

$$A_h''(\chi, \eta) = \int_{\Omega_h} \left(\sum_{i,j=1}^{N} (a_{i,j} \circ \Phi_h) D_i \chi D_j \eta + \sum_{i=1}^{N} (b_i \circ \Phi_h) D_i \chi \eta + (c \circ \Phi_h) \chi \eta \right) \tag{5.21}$$

and define the linear functional λ_h'' on functions $\chi : \Omega_h \to \mathbb{R}$ by

$$\lambda_h''(\chi) = \int_{\Omega_h} (f \circ \Phi_h) \chi. \tag{5.22}$$

Define $\check{T}_h = \{\Phi_h(\tau) : \tau \in T_h\}$. It is easily verified that $\check{T}_h$ is a finite collection of subsets of Ω with the following properties.

1. The union of the elements of $\check{T}_h$ is $\bar{\Omega}$.

2. If $\check{\tau} \in \check{T}_h$ then there exists an invertible map $\check{F}_{\check{\tau}} : T^N \to \check{\tau}$ such that $\check{\tau} = \check{F}_{\check{\tau}}(T^N)$.

3. Elements of $\check{T}_h$ meet face-to-face or not at all.

4. If $\check{\tau} \in \check{T}_h$ then $\check{F}_{\check{\tau}}$ is sufficiently smooth on T^N, $\check{F}_{\check{\tau}}^{-1}$ is sufficiently smooth on $\check{\tau}$, and $|\check{F}_{\check{\tau}}|_{(W^i_\infty(T^N))^N} \leq ch^i$ and $|\check{F}_{\check{\tau}}^{-1}|_{(W^i_\infty(\check{\tau}))^N} \leq ch^{-i}$ for all $i \in 0 : k$, where k is a sufficiently large integer.

Define $\check{S}_h = \{\chi \circ \Phi_h^{-1} : \chi \in S_h\}$. That is, $\check{S}_h$ is the set of $\check{\chi} \in C^0(\Omega)$ such that, if $\check{\tau} \in \check{T}_h$ then $\check{\chi} \circ \check{F}_{\check{\tau}} \in \Pi^{r-1}(T^N)$. Just like S_h, $\check{S}_h$ has the standard approximation, inverse, and superapproximation properties. It also respects homogeneous Dirichlet boundary conditions on Ω. The crucial difference is that, unlike the functions in S_h, which are defined on Ω_h, the functions in $\check{S}_h$ are defined on Ω. It is on Ω, and not Ω_h, that the weak problem of Equation 5.5 is posed.

Define the bilinear form $\check{A}_h$ on functions $\check{\chi}, \check{\eta} : \Omega \to \mathbb{R}$ by

$$\check{A}_h(\check{\chi}, \check{\eta}) = A''_h(\check{\chi} \circ \Phi_h, \check{\eta} \circ \Phi_h) \tag{5.23}$$

and define the linear functional $\check{\lambda}_h$ on functions $\check{\chi} : \Omega \to \mathbb{R}$ by

$$\check{\lambda}_h(\check{\chi}) = \lambda''_h(\check{\chi} \circ \Phi_h). \tag{5.24}$$

Define $\check{u}_h = u_h \circ \Phi_h^{-1}$. By Equations 5.23, 5.12, 5.24, and 5.5, it is easily seen that, if $\check{\chi} \in \check{S}_h$ and $\check{\chi} = 0$ on $\partial\Omega$, then

$$A(u - \check{u}_h, \check{\chi}) = F(\check{\chi}), \tag{5.25}$$

where

$$F = \sum_{i=1}^{6} F_i, \tag{5.26}$$

and, for $\phi : \Omega \to \mathbb{R}$,

$$F_1(\phi) = (\check{A}_h - A)(\check{u}_h, \phi), \tag{5.27}$$

$$F_2(\phi) = -(\check{\lambda}_h - \lambda)(\phi), \tag{5.28}$$

$$F_3(\phi) = (A'_h - A''_h)(u_h, \phi \circ \Phi_h), \tag{5.29}$$

$$F_4(\phi) = -(\lambda'_h - \lambda''_h)(\phi \circ \Phi_h), \tag{5.30}$$

$$F_5(\phi) = (A_h - A'_h)(u_h, \phi \circ \Phi_h), \tag{5.31}$$

$$F_6(\phi) = -(\lambda_h - \lambda'_h)(\phi \circ \Phi_h). \tag{5.32}$$

The terms F_1 and F_2 arise from transforming the weak problem from Ω to Ω_h, the terms F_3 and F_4 arise from using extensions of the data instead of composition of the data with Φ_h, and the terms F_5 and F_6 arise from numerical integration,

We now give estimates for global weighted L_∞ and W^1_∞ norms of $u - \check{u}_h$ in terms of the perturbation functional F.

Theorem 5.3. *If $k \in 0:1$, $0 \le s \le r - 2 + k$, and $x \in \Omega$ then*

$$\|u - \check{u}_h\|_{W^k_\infty(\Omega),\{x\},h,s} \le h^{r-k}\ell_{s=r-2+k,h}\|u\|_{W^r_\infty(\Omega),\{x\},h,s} + \sup_{\check{\chi} \in \check{G}^k_h} |F(\check{\chi})|, \tag{5.33}$$

where $\check{G}^k_h$ is the set of $\check{\chi} \in \check{S}_h$ with $\check{\chi} = 0$ on $\partial\Omega$ for which there exists some $v \in W^{2-k}_1(\Omega)$ with $v = 0$ on $\partial\Omega$ such that

$$\|v - \check{\chi}\|_{W^1_1(\Omega)} \le Ch^{1-k}\ell_h, \tag{5.34}$$

$$\|v\|_{W^{1-k}_1(\Omega)} \le C, \tag{5.35}$$

and

$$\|v\|_{W^{2-k}_1(\Omega)} \le C\ell_h. \tag{5.36}$$

Proof. Although the homogeneous Neumann problem is considered in the error estimates of [19, Theorems 2.1 and 3.1], an essentially identical proof can be used to furnish this result. We have used the approximation property of $\check{S}_h$ to write $h^{r-1}\|u\|_{W^r_\infty(\Omega),\{x\},s}$ in place of $\inf \|u-\check{\chi}\|_{W^1_\infty(\Omega),\{x\},h,s}$ over all $\check{\chi} \in \check{S}_h$ with $\check{\chi} = 0$ on $\partial\Omega$, as in [19, Equation 4.4].

In the $k = 0$ case, [19, Equation 2.24] shows that the left side of Equation 5.33 can be bounded by the first term on the right side of Equation 5.33 plus $|F(g_h)|$, where $g_h \in \check{S}_h$ has $g_h = 0$ on $\partial\Omega$. Furthermore, there exists some $g \in W^2_1(\Omega)$ such that, as shown in [19, Equations 2.25 and 2.26], $\|g - g_h\|_{W^1_1(\Omega)} \le Ch\ell_h$ and $\|g\|_{W^2_1(\Omega)} \le C\ell_h$. A proof that $\|g\|_{W^1_1(\Omega)} \le C$ can be easily modelled on [19, Equation 2.26]. This shows that $g_h \in \check{G}^0_h$.

In the $k = 1$ case [19, Section 3(C)] shows that the left side of Equation 5.33 can be bounded by the first term on the right side of Equation 5.33 plus $|F(\tilde{g}_h)|$, where $\tilde{g}_h \in \check{G}^1_h$. □

The following result gives estimates for the various components of the perturbation functional that appear on the right side of Equation 5.33.

Lemma 5.4. *If $\phi \in W^1_1(\Omega)$ then*

$$|F_1(\phi)| \le Ch^{m-1}\Big(\|u - \check{u}_h\|_{W^1_\infty(\Omega)} + \|u\|_{W^1_\infty(\Omega)}\Big)\|\phi\|_{W^1_1(\Omega)}. \tag{5.37}$$

If $\phi \in W^2_1(\Omega)$ then

$$|F_1(\phi)| \le C\Big(h^{m-1}\|u - \check{u}_h\|_{W^1_\infty(\Omega)}\|\phi\|_{W^1_1(\Omega)} + h^m\|u\|_{W^2_\infty(\Omega)}\|\phi\|_{W^2_1(\Omega)}\Big). \tag{5.38}$$

If $\phi \in L_1(\Omega)$ then

$$|F_2(\phi)| \le Ch^{m-1}\|f\|_{L_\infty(\Omega)}\|\phi\|_{L_1(\Omega)}. \tag{5.39}$$

If $\phi \in W^1_1(\Omega)$ then

$$|F_2(\phi)| \le Ch^m\|f\|_{W^1_\infty(\Omega)}\|\phi\|_{W^1_1(\Omega)}. \tag{5.40}$$

If $\phi \in W^1_1(\Omega)$ then

$$|F_3(\phi)| \le Ch^m\Big(\|u - \check{u}_h\|_{W^1_\infty(\Omega)} + \|u\|_{W^1_\infty(\Omega)}\Big)\|\phi\|_{W^1_1(\Omega)}. \tag{5.41}$$

If $\phi \in L_1(\Omega)$ *then*

$$|F_4(\phi)| \le Ch^m \|f\|_{W^1_\infty(\Omega)} \|\phi\|_{L_1(\Omega)}. \tag{5.42}$$

If $\check{\chi} \in \check{S}_h$ *then*

$$|F_5(\check{\chi})| \le C\Big(h^{q+1-2(r-2)} \|u - \check{u}_h\|_{W^1_\infty(\Omega)} + h^{q+3-r} \|u\|_{W^{r-1}_\infty(\Omega)}\Big) \|\check{\chi}\|_{W^1_1(\Omega)}. \tag{5.43}$$

If $\check{\chi} \in \check{S}_h$ *and* $r \ge 3$ *then*

$$\begin{aligned} |F_5(\check{\chi})| \le C\Big(&h^{q+2-2(r-2)} \|u - \check{u}_h\|_{W^1_\infty(\Omega)} + h^{q+4-r} \|u\|_{W^{r-1}_\infty(\Omega)}\Big) \\ &\times \sum_{\check{\tau} \in \check{T}_h} \|\check{\chi}\|_{W^2_1(\check{\tau})}. \end{aligned} \tag{5.44}$$

If $\check{\chi} \in \check{S}_h$ *then*

$$|F_6(\check{\chi})| \le Ch^{q+3-r} \|f\|_{W^{q+3-r}_\infty(\Omega)} \|\check{\chi}\|_{W^1_1(\Omega)}. \tag{5.45}$$

If $\check{\chi} \in \check{S}_h$ *and* $r \ge 3$ *then*

$$|F_6(\check{\chi})| \le Ch^{q+4-r} \|f\|_{W^{q+4-r}_\infty(\Omega)} \sum_{\check{\tau} \in \check{T}_h} \|\check{\chi}\|_{W^2_1(\check{\tau})}. \tag{5.46}$$

We now show how Theorems 5.2 and 5.1 follow from Theorem 5.3 and Lemma 5.4. We defer the proof of Lemma 5.4 to the next section.

First we prove Theorem 5.2. Let $\check{\chi} \in \check{G}^1_h$. That is, $\check{\chi} \in \check{S}_h$ and $\check{\chi} = 0$ on $\partial\Omega$. Furthermore, there exists some $v \in W^1_1(\Omega)$ such that

$$\|v - \check{\chi}\|_{W^1_1(\Omega)} \le C\ell_h \tag{5.47}$$

and

$$\|v\|_{W^1_1(\Omega)} \le C\ell_h. \tag{5.48}$$

Therefore,

$$\begin{aligned} \|\check{\chi}\|_{W^1_1(\Omega)} &\le \|v - \check{\chi}\|_{W^1_1(\Omega)} + \|v\|_{W^1_1(\Omega)} \\ &\le C\ell_h. \end{aligned} \tag{5.49}$$

By Equations 5.26, 5.37, 5.39, 5.41, 5.42, 5.43, 5.45, and 5.49,

$$\begin{aligned}|F(\check{\chi})| \le C\ell_h \Big(&(h^{m-1} + h^{q+1-2(r-2)})\|u - \check{u}_h\|_{W^1_\infty(\Omega)} \\ &+ h^{m-1}\Big(\|u\|_{W^1_\infty(\Omega)} + \|f\|_{W^1_\infty(\Omega)}\Big) \\ &+ h^{q+3-r}\Big(\|u\|_{W^{r-1}_\infty(\Omega)} + \|f\|_{W^{q+3-r}_\infty(\Omega)}\Big)\Big).\end{aligned} \tag{5.50}$$

Since $m - 1 \ge s + 1$ and $q + 1 - 2(r - 2) = s + 1$,

$$(h^{m-1} + h^{q+1-2(r-2)})\|u - \check{u}_h\|_{W^1_\infty(\Omega)} \le Ch^{s+1}\|u - \check{u}_h\|_{W^1_\infty(\Omega)}. \tag{5.51}$$

Since $m - 1 = q + 3 - r = r - 1 + s$,

$$h^{m-1}\|u\|_{W^1_\infty(\Omega)} + h^{q+3-r}\|u\|_{W^{r-1}_\infty(\Omega)} \le Ch^{r-1+s}\|u\|_{W^{r-1}_\infty(\Omega)} \tag{5.52}$$

and

$$h^{m-1}\|f\|_{W^1_\infty(\Omega)} + h^{q+3-r}\|f\|_{W^{q+3-r}_\infty(\Omega)} \le Ch^{r-1+s}\|f\|_{W^{r-1+s}_\infty(\Omega)}. \tag{5.53}$$

Putting together Equations 5.50, 5.51, 5.52, and 5.53, and using the fact that $\sigma_{\{x\},h} \ge Ch$ on Ω,

$$\begin{aligned}|F(\check{\chi})| \le C\ell_h \Big(&h\|u - \check{u}_h\|_{W^1_\infty(\Omega),\{x\},h,s} \\ &+ h^{r-1}\Big(\|u\|_{W^{r-1}_\infty(\Omega),\{x\},h,s} + h^s\|f\|_{W^{r-1+s}_\infty(\Omega)}\Big)\Big).\end{aligned} \tag{5.54}$$

Therefore, by Theorem 5.3,

$$\begin{aligned}\|u - \check{u}_h\|_{W^1_\infty(\Omega),\{x\},h,s} \le C\ell_h \Big(&h\|u - \check{u}_h\|_{W^1_\infty(\Omega),\{x\},h,s} \\ &+ h^{r-1}\Big(\|u\|_{W^r_\infty(\Omega),\{x\},h,s} + h^s\|f\|_{W^{r-1+s}_\infty(\Omega)}\Big)\Big).\end{aligned} \tag{5.55}$$

For h sufficiently small, $C\ell_h h \le 1/2$, so we can kick back the first term on the right side. This gives Theorem 5.2.

Next we turn to proving Theorem 5.1. Let $\check{\chi} \in \check{G}^0_h$. That is, $\check{\chi} \in \check{S}_h$ and $\check{\chi} = 0$ on $\partial\Omega$. Furthermore, there exists some $v \in W^2_1(\Omega)$ such that

$$\|v - \check{\chi}\|_{W^1_1(\Omega)} \le Ch\ell_h, \tag{5.56}$$

$$\|v\|_{W_1^1(\Omega)} \le C, \tag{5.57}$$

and

$$\|v\|_{W_1^2(\Omega)} \le C\ell_h. \tag{5.58}$$

Therefore,

$$\begin{aligned} \|\check{\chi}\|_{W_1^1(\Omega)} &\le \|v - \check{\chi}\|_{W_1^1(\Omega)} + \|v\|_{W_1^1(\Omega)} \\ &\le C. \end{aligned} \tag{5.59}$$

By the approximation property of $\check{S}_h$, there exists some $\check{\eta} \in \check{S}_h$ such that, if $i \in 0:2$ then

$$\sum_{\check{\tau} \in \check{T}_h} |v - \check{\eta}|_{W_1^i(\check{\tau})} \le Ch^{2-i}\|v\|_{W_1^2(\Omega)}. \tag{5.60}$$

By the inverse property of $\check{S}_h$,

$$\begin{aligned} \sum_{\check{\tau} \in \check{T}_h} \|\check{\chi} - \check{\eta}\|_{W_1^2(\check{\tau})} &\le Ch^{-1}\|\check{\chi} - \check{\eta}\|_{W_1^1(\Omega)} \\ &\le Ch^{-1}\Big(\|v - \check{\chi}\|_{W_1^1(\Omega)} + \|v - \check{\eta}\|_{W_1^1(\Omega)}\Big). \end{aligned} \tag{5.61}$$

Putting together Equations 5.61, 5.56, 5.60 and 5.58,

$$\begin{aligned} \sum_{\check{\tau} \in \check{T}_h} \|\check{\chi}\|_{W_1^2(\check{\tau})} &\le \sum_{\check{\tau} \in \check{T}_h} \Big(\|\check{\chi} - \check{\eta}\|_{W_1^2(\check{\tau})} + \|v - \check{\eta}\|_{W_1^2(\check{\tau})} + \|v\|_{W_1^2(\check{\tau})}\Big) \\ &\le C\ell_h. \end{aligned} \tag{5.62}$$

Suppose that $r \ge 3$. By Equations 5.26, 5.37, 5.38, 5.40, 5.41, 5.42, 5.44, 5.46, 5.56, 5.57, 5.58, 5.59, and 5.62,

$$\begin{aligned} |F(\check{\chi})| &\le |F_1(\check{\chi} - v)| + |F_1(v)| + \sum_{i=2}^{6} |F_i(\check{\chi})| \\ &\le C\bigg((h^{m-1} + h^{q+1-2(r-2)})\|u - \check{u}_h\|_{W_\infty^1(\Omega)} \\ &\quad + h^m\Big(\ell_h\|u\|_{W_\infty^2(\Omega)} + \|f\|_{W_\infty^1(\Omega)}\Big) \\ &\quad + h^{q+4-r}\ell_h\Big(\|u\|_{W_\infty^{r-1}(\Omega)} + \|f\|_{W_\infty^{q+4-r}(\Omega)}\Big)\bigg). \end{aligned} \tag{5.63}$$

Since $m = q + 4 - r = r + s$,

$$h^m \|u\|_{W^2_\infty(\Omega)} + h^{q+4-r} \|u\|_{W^{r-1}_\infty(\Omega)} \leq C h^{r+s} \|u\|_{W^{r-1}_\infty(\Omega)} \tag{5.64}$$

and

$$h^m \|f\|_{W^1_\infty(\Omega)} + h^{q+4-r} \|f\|_{W^{q+4-r}_\infty(\Omega)} \leq C h^{r+s} \|f\|_{W^{r+s}_\infty(\Omega)}. \tag{5.65}$$

Putting together Equations 5.63, 5.51, 5.64, and 5.65, and using the fact that $\sigma_{\{x\},h} \geq Ch$ on Ω,

$$\begin{aligned} |F(\check{\chi})| \leq Ch\Big(&\|u - \check{u}_h\|_{W^1_\infty(\Omega),\{x\},h,s} \\ &+ h^{r-1} \ell_h \Big(\|u\|_{W^{r-1}_\infty(\Omega),\{x\},h,s} + h^s \|f\|_{W^{r+s}_\infty(\Omega)} \Big) \Big). \end{aligned} \tag{5.66}$$

Therefore, by Theorem 5.3,

$$\begin{aligned} \|u - \check{u}_h\|_{L_\infty(\Omega),\{x\},h,s} \leq Ch\Big(&\|u - \check{u}_h\|_{W^1_\infty(\Omega),\{x\},h,s} \\ &+ h^{r-1} \ell_h \Big(\|u\|_{W^{r}_\infty(\Omega),\{x\},h,s} + h^s \|f\|_{W^{r+s}_\infty(\Omega)} \Big) \Big). \end{aligned} \tag{5.67}$$

Using Theorem 5.2 to estimate the first term on the right side, we obtain Theorem 5.1.

Now suppose that $r = 2$. In this case, $q = 1 + s$. By Equations 5.26, 5.37, 5.38, 5.40, 5.41, 5.42, 5.43, 5.45, 5.56, 5.57, 5.58, and 5.59,

$$\begin{aligned} |F(\check{\chi})| &\leq |F_1(\check{\chi} - v)| + |F_1(v)| + \sum_{i=2}^{6} |F_i(\check{\chi})| \\ &\leq C\Big((h^{m-1} + h^{q+1-2(r-2)}) \|u - \check{u}_h\|_{W^1_\infty(\Omega)} \\ &\quad + h^m \Big(\ell_h \|u\|_{W^2_\infty(\Omega)} + \|f\|_{W^1_\infty(\Omega)} \Big) \\ &\quad + h^{q+3-r} \Big(\|u\|_{W^{r-1}_\infty(\Omega)} + \|f\|_{W^{q+3-r}_\infty(\Omega)} \Big) \Big). \end{aligned} \tag{5.68}$$

Since $m = q + 3 - r = r + s$,

$$h^m \|u\|_{W^2_\infty(\Omega)} + h^{q+3-r} \|u\|_{W^{r-1}_\infty(\Omega)} \leq C h^{r+s} \|u\|_{W^{r}_\infty(\Omega)} \tag{5.69}$$

and

$$h^m \|f\|_{W^1_\infty(\Omega)} + h^{q+3-r} \|f\|_{W^{q+3-r}_\infty(\Omega)} \leq C h^{r+s} \|f\|_{W^{r+s}_\infty(\Omega)}. \tag{5.70}$$

Putting together Equations 5.68, 5.51, 5.69, and 5.70, and using the fact that $\sigma_{\{x\},h} \geq Ch$ on Ω, we obtain Equation 5.66. Theorem 5.1 follows from this and Theorems 5.3 and 5.2, as in the $r \geq 3$ case.

5.5 Analysis of Perturbation Terms

5.5.1 Terms due to the Mapping Φ_h

We begin with a simple result in which the difference of two products of the same length is expressed in terms of the differences of corresponding terms.

Proposition 5.5. *If $n \geq 1$ is an integer and, for $i \in 1:n$, $a_i, b_i \in \mathbb{R}$, then*

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{j=1}^{n} \Big(\prod_{i=1}^{j-1} a_i \Big) (a_j - b_j) \Big(\prod_{i=j+1}^{n} b_i \Big). \tag{5.71}$$

Proof. The proof is by induction on n. The $n = 1$ case is trivial. The $n = 2$ case,

$$\begin{aligned} a_1 a_2 - b_1 b_2 &= a_1 a_2 - a_1 b_2 + a_1 b_2 - b_1 b_2 \\ &= a_1(a_2 - b_2) + (a_1 - b_1) b_2, \end{aligned} \tag{5.72}$$

is used to prove the induction step. Assuming the proposition holds for some arbitrary integer $n \geq 1$, we find that

$$\begin{aligned}
\prod_{i=1}^{n+1} a_i - \prod_{i=1}^{n+1} b_i &= \Big(\prod_{i=1}^{n} a_i\Big) a_{n+1} - \Big(\prod_{i=1}^{n} b_i\Big) b_{n+1} \\
&= \Big(\prod_{i=1}^{n} a_i\Big)(a_{n+1} - b_{n+1}) + \Big(\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i\Big) b_{n+1} \\
&= \Big(\prod_{i=1}^{n} a_i\Big)(a_{n+1} - b_{n+1})\Big(\prod_{i=n+2}^{n+1} b_i\Big) \\
&\qquad + \sum_{j=1}^{n} \Big(\prod_{i=1}^{j-1} a_i\Big)(a_j - b_j)\Big(\prod_{i=j+1}^{n} b_i\Big) b_{n+1} \\
&= \sum_{j=1}^{n+1} \Big(\prod_{i=1}^{j-1} a_i\Big)(a_j - b_j)\Big(\prod_{i=j+1}^{n+1} b_i\Big),
\end{aligned} \tag{5.73}$$

which demonstrates the $n + 1$ case. □

By definition of the matrix determinant and Proposition 5.5,

$$\begin{aligned}
\det D\Phi_h^{-1} - 1 &= \det D\Phi_h^{-1} - \det DI \\
&= \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \Big(\prod_{i=1}^{N} D_{\sigma_i} (\Phi_h^{-1})_i - \prod_{i=1}^{N} D_{\sigma_i} I_i\Big) \\
&= \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \\
&\qquad \times \sum_{i=1}^{N} \Big(\prod_{j=1}^{i-1} D_{\sigma_j} (\Phi_h^{-1})_j\Big) D_{\sigma_i} (\Phi_h^{-1} - I)_i \Big(\prod_{j=i+1}^{N} D_{\sigma_j} I_j\Big),
\end{aligned} \tag{5.74}$$

where S_N is the set of permutations of $1 : N$, considered as elements of $(1 : N)^N$, and sgn is the signature map on S_N. Therefore, by Equation 5.7,

$$\| \det D\Phi_h^{-1} - 1 \|_{L_\infty(\Omega)} \leq C h^{m-1}. \tag{5.75}$$

This is given in [18, Proposition 3(ii)] for the particular Φ_h constructed in [18, Section 5.1]. We have derived it here in order to make it clear that it follows from the assumption of Equation 5.7 and is not exclusive to any particular Φ_h.

First suppose that $v \in W^1_\infty(\Omega)$ and $\phi \in W^1_1(\Omega)$. Observe that

$$\check{A}_h(v,\phi) = \int_\Omega \biggl(\sum_{i,j=1}^N (\check{a}_{i,j})_h D_i v D_j \phi + \sum_{i=1}^N (\check{b}_i)_h D_i v \phi + \check{c}_h v \phi \biggr), \tag{5.76}$$

where

$$(\check{a}_{i,j})_h = \sum_{k,\ell=1}^N a_{k,\ell}((D_k(\Phi_h)_i D_\ell(\Phi_h)_j) \circ \Phi_h^{-1}) \det D\Phi_h^{-1}, \tag{5.77}$$

$$(\check{b}_i)_h = \sum_{k=1}^N b_k(D_k(\Phi_h)_i \circ \Phi_h^{-1}) \det D\Phi_h^{-1}, \tag{5.78}$$

and

$$\check{c}_h = c \det D\Phi_h^{-1}. \tag{5.79}$$

By Equations 5.76 and 5.1,

$$\begin{aligned}(\check{A}_h - A)(v,\phi) = \int_\Omega \biggl(&\sum_{i,j=1}^N ((\check{a}_{i,j})_h - a_{i,j}) D_i v D_j \phi \\ &+ \sum_{i=1}^N ((\check{b}_i)_h - b_i) D_i v \phi \\ &+ (\check{c}_h - c) v \phi \biggr).\end{aligned} \tag{5.80}$$

Notice that

$$\begin{aligned}(\check{a}_{i,j})_h - a_{i,j} &= \sum_{k,\ell=1}^N a_{k,\ell}((D_k(\Phi_h)_i D_\ell(\Phi_h)_j) \circ \Phi_h^{-1}) \det D\Phi_h^{-1} - a_{i,j} \\ &= \sum_{k,\ell=1}^N a_{k,\ell}\Bigl(((D_k(\Phi_h)_i D_\ell(\Phi_h)_j) \circ \Phi_h^{-1}) \det D\Phi_h^{-1} - \delta_{i,k}\delta_{j,\ell}\Bigr) \\ &= \sum_{k,\ell=1}^N a_{k,\ell}\Bigl(((D_k(\Phi_h - I)_i) D_\ell(\Phi_h)_j) \circ \Phi_h^{-1}) \det D\Phi_h^{-1} \\ &\quad + \delta_{i,k}((D_\ell(\Phi_h - I)_j) \circ \Phi_h^{-1}) \det D\Phi_h^{-1} \\ &\quad + \delta_{i,k}\delta_{j,\ell}(\det D\Phi_h^{-1} - 1)\Bigr),\end{aligned} \tag{5.81}$$

$$
\begin{aligned}
(\check{b}_i)_h - b_i &= \sum_{k=1}^{N} b_k (D_k(\Phi_h)_i \circ \Phi_h^{-1}) \det D\Phi_h^{-1} - b_i \\
&= \sum_{k=1}^{N} b_k \Big((D_k(\Phi_h)_i \circ \Phi_h^{-1}) \det D\Phi_h^{-1} - \delta_{i,k} \Big) \\
&= \sum_{k=1}^{N} b_k \Big(((D_k(\Phi_h - I)_i) \circ \Phi_h^{-1}) \det D\Phi_h^{-1} \\
&\qquad + \delta_{i,k} (\det D\Phi_h^{-1} - 1) \Big),
\end{aligned} \tag{5.82}
$$

and

$$
\check{c}_h - c = c(\det D\Phi_h^{-1} - 1). \tag{5.83}
$$

Using Equations 5.80, 5.81, 5.82, 5.83, 5.7 and 5.75, we see that

$$
|(\check{A}_h - A)(v, \phi)| \leq C h^{m-1} \|v\|_{W^1_\infty(\Omega)} \|\phi\|_{W^1_1(\Omega)}. \tag{5.84}
$$

In particular, by Equation 5.27,

$$
|F_1(\phi)| \leq C h^{m-1} \|\check{u}_h\|_{W^1_\infty(\Omega)} \|\phi\|_{W^1_1(\Omega)}, \tag{5.85}
$$

from which Equation 5.37 follows.

Now suppose that $\phi \in L_1(\Omega)$. Observe that

$$
\check{\lambda}_h(\phi) = \int_\Omega \check{f}_h \phi, \tag{5.86}
$$

where

$$
\check{f}_h = f \det D\Phi_h^{-1}. \tag{5.87}
$$

By Equations 5.86 and 5.4,

$$
(\check{\lambda}_h - \lambda)(\phi) = \int_\Omega (\check{f}_h - f)\phi. \tag{5.88}
$$

Notice that

$$
\check{f}_h - f = f(\det D\Phi_h^{-1} - 1). \tag{5.89}
$$

Using Equations 5.88, 5.89, and 5.75, we see that

$$
|(\check{\lambda}_h - \lambda)(\phi)| \leq C h^{m-1} \|f\|_{L_\infty(\Omega)}. \tag{5.90}
$$

Equation 5.39 follows from this and Equation 5.28.

Equations 5.38 and 5.40 are more difficult to prove than Equations 5.37 and 5.39. The basic idea is that we use integration by parts to transfer a derivative from $\Phi_h - I$ or $\Phi_h^{-1} - I$ to ϕ.

The following result will imply Equations 5.38 and 5.40.

Proposition 5.6. *Suppose that $K \in W_1^1(\Omega)$ and $i, j, k, \ell \in 1 : N$. Let*

$$J_1 = \int_\Omega K(D_i(\Phi_h - I)_j \circ \Phi_h^{-1}) \det D\Phi_h^{-1}, \tag{5.91}$$

$$J_2 = \int_\Omega K\Big(D_i(\Phi_h - I)_j D_k(\Phi_h)_\ell \circ \Phi_h^{-1}\Big) \det D\Phi_h^{-1}, \tag{5.92}$$

$$J_3 = \int_\Omega K(\det D\Phi_h^{-1} - 1). \tag{5.93}$$

Then $|J_1|, |J_2|, |J_3| \leq Ch^m \|K\|_{W_1^1(\Omega)}$.

Before proving this, we see how Equations 5.38 and 5.40 follow from it.

First suppose that $\phi \in W_1^2(\Omega)$. By Equation 5.81 and Proposition 5.6,

$$|\int_\Omega ((\breve{a}_{i,j})_h - a_{i,j}) D_i u D_j \phi| \leq Ch^m \|u\|_{W_\infty^2(\Omega)} \|\phi\|_{W_1^2(\Omega)}. \tag{5.94}$$

By Equation 5.82 and Proposition 5.6,

$$|\int_\Omega ((\breve{b}_i)_h - b_i) D_i u \phi| \leq Ch^m \|u\|_{W_\infty^2(\Omega)} \|\phi\|_{W_1^1(\Omega)}. \tag{5.95}$$

By Equations 5.83 and Proposition 5.6,

$$|\int_\Omega (\breve{c}_h - c) u \phi| \leq Ch^m \|u\|_{W_\infty^1(\Omega)} \|\phi\|_{W_1^1(\Omega)}. \tag{5.96}$$

Putting together Equations 5.80, 5.94, 5.95, and 5.96, we see that

$$|(\breve{A}_h - A)(u, \phi)| \leq Ch^m \|u\|_{W_\infty^2(\Omega)} \|\phi\|_{W_1^2(\Omega)}. \tag{5.97}$$

It is not possible to substitute $\breve{u}_h$ for u here, because $\breve{u}_h$ does not necessarily have two derivatives on Ω. However, from Equation 5.84, we know that

$$|(\breve{A}_h - A)(u - \breve{u}_h, \phi)| \leq Ch^{m-1} \|u - \breve{u}_h\|_{W_\infty^1(\Omega)} \|\phi\|_{W_1^1(\Omega)}. \tag{5.98}$$

Equation 5.38 follows from Equations 5.27, 5.97, and 5.98.

Now suppose that $\phi \in W_1^1(\Omega)$. By Equation 5.89 and Proposition 5.6,

$$|\int_\Omega (\check{f}_h - f)\phi| \le Ch^m \|f\|_{W^1_\infty(\Omega)} \|\phi\|_{W_1^1(\Omega)}. \tag{5.99}$$

Equation 5.40 follows from Equations 5.28, 5.88, and 5.99.

Finally we turn to the proof of Proposition 5.6. First observe that, by integration by parts,

$$\begin{aligned}\int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j = \int_{\partial\Omega_h} &(K \circ \Phi_h)(\Phi_h - I)_j (\nu_{\Omega_h})_i \, dS \\ &- \int_{\Omega_h} D_i(K \circ \Phi_h)(\Phi_h - I)_j.\end{aligned} \tag{5.100}$$

Therefore, by Equation 5.6 and the trace inequality,

$$\begin{aligned}|\int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j| &\le Ch^m \Big(\|K \circ \Phi_h\|_{L_1(\partial\Omega_h)} + |K \circ \Phi_h|_{W_1^1(\Omega_h)} \Big) \\ &\le Ch^m \|K\|_{W_1^1(\Omega)}.\end{aligned} \tag{5.101}$$

By the change of variables formula,

$$J_1 = \int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j, \tag{5.102}$$

so Equation 5.101 furnishes the bound on J_1.

By the change of variables formula,

$$\begin{aligned}J_2 &= \int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j D_k(\Phi_h)_\ell \\ &= \int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j \Big(D_k(\Phi_h - I)_\ell + D_k I_\ell \Big) \\ &= \int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j D_k(\Phi_h - I)_\ell \\ &\quad + \delta_{k,\ell} \int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j.\end{aligned} \tag{5.103}$$

By Equation 5.6,

$$\begin{aligned}|\int_{\Omega_h} (K \circ \Phi_h) D_i(\Phi_h - I)_j D_k(\Phi_h - I)_\ell| &\le Ch^{2(m-1)} \|K \circ \Phi_h\|_{L_1(\Omega_h)} \\ &\le Ch^{2(m-1)} \|K\|_{L_1(\Omega)}.\end{aligned} \tag{5.104}$$

Equations 5.103, 5.104, and 5.101 give the bound on J_2.

All that remains is to prove the bound on J_3. A straightforward integration by parts, using Equation 5.74 to represent $\det D\Phi_h^{-1} - 1$, would involve transferring a derivative from $\Phi_h^{-1} - I$ to Φ_h^{-1}. This is not possible because Φ_h^{-1} does not necessarily have two derivatives on Ω. A more intricate representation of $\det D\Phi_h^{-1} - 1$ is needed.

For $\sigma \in S_N$ and $i \in 1 : N$, let

$$P_{\sigma,i} = \operatorname{sgn}(\sigma) D_{\sigma_i}(\Phi_h^{-1} - I)_i \Big(\prod_{j=i+1}^{N} D_{\sigma_j} I_j \Big), \tag{5.105}$$

$$Q_{\sigma,i} = \prod_{j=1}^{i-1} D_{\sigma_j} I_j, \tag{5.106}$$

and

$$R_{\sigma,i} = \sum_{j=1}^{i-1} \Big(\prod_{k=1}^{j-1} D_{\sigma_k}(\Phi_h^{-1})_k \Big) D_{\sigma_j}(\Phi_h^{-1} - I)_j \Big(\prod_{k=j+1}^{i-1} D_{\sigma_k} I_k \Big). \tag{5.107}$$

Obviously

$$\prod_{j=1}^{i-1} D_{\sigma_j}(\Phi_h^{-1})_j = Q_{\sigma,i} + \Big(\prod_{j=1}^{i-1} D_{\sigma_j}(\Phi_h^{-1})_j - \prod_{j=1}^{i-1} D_{\sigma_j} I_j \Big). \tag{5.108}$$

By Proposition 5.5,

$$\prod_{j=1}^{i-1} D_{\sigma_j}(\Phi_h^{-1})_j - \prod_{j=1}^{i-1} D_{\sigma_j} I_j = R_{\sigma,i}. \tag{5.109}$$

Putting together Equations 5.74, 5.105, 5.108, and 5.109, we see that

$$\det D\Phi^{-1} - 1 = \sum_{\sigma \in S_N} \sum_{i=1}^{N} P_{\sigma,i}(Q_{\sigma,i} + R_{\sigma,i}). \tag{5.110}$$

Observe that

$$P_{\sigma,i} Q_{\sigma,i} = \begin{cases} D_i(\Phi_h^{-1} - I)_i, & \text{if } \sigma = I \\ 0, & \text{otherwise.} \end{cases} \tag{5.111}$$

By integration by parts,

$$\begin{aligned} \int_\Omega K D_i(\Phi_h^{-1} - I)_i = \int_{\partial\Omega} & K(\Phi_h^{-1} - I)_i (\nu_\Omega)_i \, dS \\ & - \int_\Omega D_i K (\Phi_h^{-1} - I)_i. \end{aligned} \tag{5.112}$$

By Equations 5.111, 5.112, and 5.7, along with the trace inequality,

$$\begin{aligned} |\int_\Omega K P_{\sigma,i} Q_{\sigma,i}| &\le Ch^m \Big(\|K\|_{L_1(\partial\Omega)} + |K|_{W_1^1(\Omega)} \Big) \\ &\le Ch^m \|K\|_{W_1^1(\Omega)}. \end{aligned} \tag{5.113}$$

By Equation 5.7,

$$|\int_\Omega K P_{\sigma,i} R_{\sigma,i}| \le Ch^{2(m-1)} \|K\|_{L_1(\Omega)}. \tag{5.114}$$

The bound on J_3 follows from Equations 5.110, 5.113, and 5.114.

5.5.2 Terms due to Using Extended and not Mapped Data

These terms are the most straightforward to estimate.

Assume first that $\phi \in W_1^1(\Omega)$. Then

$$\begin{aligned} (A'_h - A''_h)(u_h, \phi \circ \Phi_h) = \int_{\Omega_h} \Bigg(&\sum_{i,j=1}^N (a_{i,j} \circ \Phi_h - \bar{a}_{i,j}) D_i u_h D_j (\phi \circ \Phi_h) \\ &+ \sum_{i=1}^N (b_i \circ \Phi_h - \bar{b}_i) D_i u_h (\phi \circ \Phi_h) \\ &+ (c \circ \Phi_h - \bar{c}) u_h (\phi \circ \Phi_h) \Bigg). \end{aligned} \tag{5.115}$$

Since

$$a_{i,j} \circ \Phi_h - \bar{a}_{i,j} = \bar{a}_{i,j} \circ \Phi_h - \bar{a}_{i,j} \circ I, \tag{5.116}$$

we see by the mean value theorem and Equation 5.6 that

$$\|a_{i,j} \circ \Phi_h - \bar{a}_{i,j}\|_{L_\infty(\Omega_h)} \le Ch^m. \tag{5.117}$$

The same argument establishes that

$$\|b_i \circ \Phi_h - \bar{b}_i\|_{L_\infty(\Omega_h)} \le Ch^m \tag{5.118}$$

and

$$\|c \circ \Phi_h - \bar{c}\|_{L_\infty(\Omega_h)} \le Ch^m. \tag{5.119}$$

By Equations 5.115, 5.117, 5.118, and 5.119,

$$\begin{aligned}|(A'_h - A''_h)(u_h, \phi\circ\Phi_h)| &\le Ch^m \|u_h\|_{W^1_\infty(\Omega_h)} \|\phi\circ\Phi_h\|_{W^1_1(\Omega_h)} \\ &\le Ch^m \|\check{u}_h\|_{W^1_\infty(\Omega)} \|\phi\|_{W^1_1(\Omega)}.\end{aligned} \tag{5.120}$$

Equation 5.41 follows from this and Equation 5.29.

Next assume only that $\phi \in L_1(\Omega)$. Then

$$(\lambda'_h - \lambda''_h)(\phi\circ\Phi_h) = \int_{\Omega_h} (f\circ\Phi_h - \bar{f})(\phi\circ\Phi_h). \tag{5.121}$$

Since

$$f\circ\Phi_h - \bar{f} = \bar{f}\circ\Phi_h - \bar{f}\circ I, \tag{5.122}$$

we see by the mean value theorem and Equation 5.6 that

$$\|f\circ\Phi_h - \bar{f}\|_{L_\infty(\Omega_h)} \le Ch^m |\bar{f}|_{W^1_\infty(\Omega_h)}. \tag{5.123}$$

By Equations 5.121 and 5.123,

$$\begin{aligned}|(\lambda'_h - \lambda''_h)(\phi\circ\Phi_h)| &\le Ch^m |\bar{f}|_{W^1_\infty(\Omega_h)} \|\phi\circ\Phi_h\|_{L_1(\Omega_h)} \\ &\le Ch^m \|f\|_{W^1_\infty(\Omega)} \|\phi\|_{L_1(\Omega)}.\end{aligned} \tag{5.124}$$

Equation 5.42 follows from this and Equation 5.30.

5.5.3 Terms due to Quadrature Error

We estimate the quadrature error on each element and then sum over all the elements. Since all quadratures ultimately take place on the unit simplex, we begin by defining $\hat{E} \in (C^0(T^N))'$ by

$$\hat{E}(\hat{v}) = \int_{T^N} \hat{v} - \hat{Q}(\hat{v}). \tag{5.125}$$

For $\tau \in T_h$, define $E_\tau \in (C^0(\tau))'$ by

$$\begin{aligned}E_\tau(v) &= \int_\tau v - Q_\tau(v) \\ &= \int_{T^N} (v\circ F_\tau)\det DF_\tau - \hat{Q}\Big((v\circ F_\tau)\det DF_\tau\Big) \\ &= \hat{E}\Big((v\circ F_\tau)\det DF_\tau\Big).\end{aligned} \tag{5.126}$$

The following is the central result we will use to estimate quadrature error.

Proposition 5.7. *If $\tau \in T_h$, $k \in 0:q$, $\hat{\chi} \in \Pi^k(T^N)$, and $v \in W_\infty^{q-k+1}(\tau)$ then, with $\chi = \hat{\chi} \circ F_\tau^{-1}$,*

$$|E_\tau(v\chi)| \leq C h^{q-k+1+N} \|v\|_{W_\infty^{q-k+1}(\tau)} \|\hat{\chi}\|_{L_1(T^N)}. \tag{5.127}$$

Proof. Define $\tilde{E} \in (C^0(T^N))'$ by $\tilde{E}(\hat{v}) = \hat{E}(\hat{v}\hat{\chi})$. If $\hat{v} \in W_\infty^{q-k+1}(T^N)$ then, using the facts that $\hat{Q} \in (L_\infty(T^N))'$ and all norms on the finite-dimensional vector space $\Pi^k(T^N)$ are equivalent,

$$\begin{aligned} |\tilde{E}(\hat{v})| &\leq C \|\hat{v}\|_{L_\infty(T^N)} \|\hat{\chi}\|_{L_\infty(T^N)} \\ &\leq C \|\hat{v}\|_{W_\infty^{q-k+1}(T^N)} \|\hat{\chi}\|_{L_1(T^N)}. \end{aligned} \tag{5.128}$$

That is, $\|\tilde{E}\|_{(W_\infty^{q-k+1}(T^N))'} \leq C \|\hat{\chi}\|_{L_1(T^N)}$. By Equation 5.8, if $\hat{v} \in \Pi^{q-k}(T^N)$ then $\hat{E}(\hat{v}\hat{\chi}) = 0$. Therefore, by the Bramble-Hilbert lemma of [7, Theorem 28.1], if $\hat{v} \in W_\infty^{q-k+1}(T^N)$ then

$$|\tilde{E}(\hat{v})| \leq C |\hat{v}|_{W_\infty^{q-k+1}(T^N)} \|\hat{\chi}\|_{L_1(T^N)}. \tag{5.129}$$

By Equation 5.126,

$$\begin{aligned} |E_\tau(v\chi)| &= |\tilde{E}\Big((v \circ F_\tau) \det DF_\tau\Big)| \\ &\leq C |(v \circ F_\tau) \det DF_\tau|_{W_\infty^{q-k+1}(T^N)} \|\hat{\chi}\|_{L_1(T^N)}. \end{aligned} \tag{5.130}$$

By a scaling inequality,

$$\begin{aligned} |(v \circ F_\tau) \det DF_\tau|_{W_\infty^{q-k+1}(T^N)} \leq C h^{q-k+1} &\|v\|_{W_\infty^{q-k+1}(\tau)} \\ &\times \|\det DF_\tau \circ F_\tau^{-1}\|_{W_\infty^{q-k+1}(\tau)}. \end{aligned} \tag{5.131}$$

By a scaling inequality and the assumptions on F_τ and F_τ^{-1},

$$\|\det DF_\tau \circ F_\tau^{-1}\|_{W_\infty^{q-k+1}(\tau)} \leq C h^N. \tag{5.132}$$

The proposition follows by combining Equations 5.130, 5.131, and 5.132. $\square$

Let $\chi \in S_h$ be fixed throughout the rest of this subsection.

First we prove Equations 5.43 and 5.44. Let $\tau \in T_h$. By Proposition 5.7,

$$\begin{aligned} |E_\tau\Big(\sum_{i,j=1}^N \bar{a}_{i,j} D_i u_h D_j \chi + \sum_{i=1}^N \bar{b}_i D_i u_h \chi + \bar{c} u_h \chi\Big)| \leq C h^{q+1+N} \|u_h\|_{W_\infty^{q+2}(\tau)} \\ \times \|\chi\|_{W_\infty^{q+2}(\tau)}. \end{aligned} \tag{5.133}$$

By the element approximation property of S_h, there exists some $\eta \in S_h$ such that, for $i \in 0 : r-1$,

$$|u \circ \Phi_h - \eta|_{W_\infty^i(\tau)} \leq C h^{r-1-i} \|u \circ \Phi_h\|_{W_\infty^{r-1}(\tau)}. \tag{5.134}$$

By the element inverse property of S_h,

$$\begin{aligned} \|u_h\|_{W_\infty^{q+2}(\tau)} &\leq C \|u_h\|_{W_\infty^{r-1}(\tau)} \\ &\leq C\Big(\|u_h - \eta\|_{W_\infty^{r-1}(\tau)} \\ &\quad + \|u \circ \Phi_h - \eta\|_{W_\infty^{r-1}(\tau)} \\ &\quad + \|u \circ \Phi_h\|_{W_\infty^{r-1}(\tau)}\Big). \end{aligned} \tag{5.135}$$

Again using the element inverse property of S_h,

$$\begin{aligned} \|u_h - \eta\|_{W_\infty^{r-1}(\tau)} &\leq C h^{-(r-2)} \|u_h - \eta\|_{W_\infty^1(\tau)} \\ &\leq C h^{-(r-2)}\Big(\|u \circ \Phi_h - u_h\|_{W_\infty^1(\tau)} + \|u \circ \Phi_h - \eta\|_{W_\infty^1(\tau)}\Big). \end{aligned} \tag{5.136}$$

Putting together Equations 5.135, 5.136, and 5.134, we see that

$$\|u_h\|_{W_\infty^{q+2}(\tau)} \leq C\Big(h^{-(r-2)} \|u \circ \Phi_h - u_h\|_{W_\infty^1(\tau)} + \|u \circ \Phi_h\|_{W_\infty^{r-1}(\tau)}\Big). \tag{5.137}$$

Again using the element inverse property of S_h,

$$\|\chi\|_{W_\infty^{q+1}(\tau)} \leq C \|\chi\|_{W_\infty^{r-1}(\tau)}, \tag{5.138}$$

$$\|\chi\|_{W_\infty^{r-1}(\tau)} \leq C h^{-(r-2)-N} \|\chi\|_{W_1^1(\tau)}, \tag{5.139}$$

and, for $r \geq 3$,

$$\|\chi\|_{W_\infty^{r-1}(\tau)} \leq C h^{-(r-3)-N} \|\chi\|_{W_1^2(\tau)}. \tag{5.140}$$

Putting together Equations 5.17, 5.10, 5.126, 5.133, 5.137, 5.138, and 5.139,

$$\begin{aligned}|(A_h' - A_h)(u_h, \chi)| &\le \sum_{\tau \in T_h} |E_\tau\Big(\sum_{i,j=1}^{N} \bar{a}_{i,j} D_i u_h D_j \chi + \sum_{i=1}^{N} \bar{b}_i D_i u_h \chi + \bar{c} u_h \chi\Big)| \\ &\le C\Big(h^{q+1-2(r-2)} \|u \circ \Phi_h - u_h\|_{W^1_\infty(\Omega_h)} \\ &\quad + h^{q+3-r} \|u \circ \Phi_h\|_{W^{r-1}_\infty(\Omega_h)}\Big) \|\chi\|_{W^1_1(\Omega_h)}.\end{aligned} \tag{5.141}$$

Together with Equation 5.31, this implies Equation 5.43. If $r \ge 3$ then, using Equation 5.140 instead of 5.139 in the above, we find that

$$\begin{aligned}|(A_h' - A_h)(u_h, \chi)| \le C\Big(&h^{q+2-2(r-2)} \|u \circ \Phi_h - u_h\|_{W^1_\infty(\Omega_h)} \\ &+ h^{q+4-r} \|u \circ \Phi_h\|_{W^{r-1}_\infty(\Omega_h)}\Big) \sum_{\tau \in T_h} \|\chi\|_{W^2_1(\tau)}.\end{aligned} \tag{5.142}$$

Together with Equation 5.31, this implies Equation 5.44.

Now we prove Equations 5.45 and 5.46. Let $\tau \in T_h$ and $k \in 1:2$. Our first goal is to show that

$$|E_\tau(\bar{f}\chi)| \le C h^{q+2+k-r} \|\bar{f}\|_{W^{q+2+k-r}_\infty(\tau)} \|\chi\|_{W^k_1(\tau)}. \tag{5.143}$$

First we handle the cases $r \ge 4$ and the case where $r = 3$ and $k = 1$. Let $\hat{\chi} = \chi \circ F_\tau$. By definition of S_h, $\hat{\chi} \in \Pi^{r-1}(T^N)$. By the Bramble-Hilbert lemma, there exists some $\hat{\eta} \in \Pi^{k-1}(T^N)$ such that

$$\|\hat{\chi} - \hat{\eta}\|_{W^k_1(T^N)} \le C |\hat{\chi}|_{W^k_1(T^N)}. \tag{5.144}$$

Let $\eta = \hat{\eta} \circ F_\tau^{-1}$. Since $k - 1 \le r - 1$, we have that $\hat{\chi} - \hat{\eta} \in \Pi^{r-1}(T^N)$. Also notice that $r - 1 \le q$. Therefore, by Proposition 5.7,

$$|E_\tau(\bar{f}(\chi - \eta))| \le C h^{q+2-r+N} \|\bar{f}\|_{W^{q+2-r}_\infty(\tau)} \|\hat{\chi} - \hat{\eta}\|_{L_1(T^N)}. \tag{5.145}$$

By Equation 5.144 and a scaling inequality,

$$\|\hat{\chi} - \hat{\eta}\|_{L_1(T^N)} \le C h^{k-N} \|\chi\|_{W^k_1(\tau)}. \tag{5.146}$$

Combining Equations 5.145 and 5.146, we see that

$$|E_\tau(\bar{f}(\chi - \eta))| \le Ch^{q+2+k-r} \|\bar{f}\|_{W_\infty^{q+2-r}(\tau)} \|\chi\|_{W_1^k(\tau)}. \tag{5.147}$$

Since $k-1 \le r-k-1$, we have that $\hat{\eta} \in \Pi^{r-k-1}(T^N)$. Also notice that $r-k-1 \le q$. Therefore, by Proposition 5.7,

$$|E_\tau(\bar{f}\eta)| \le Ch^{q+2+k-r+N} \|\bar{f}\|_{W_\infty^{q+2+k-r}(\tau)} \|\hat{\eta}\|_{L_1(T^N)}. \tag{5.148}$$

By Equation 5.144 and a scaling inequality,

$$\begin{aligned} \|\hat{\eta}\|_{L_1(T^N)} &\le \|\hat{\chi} - \hat{\eta}\|_{L_1(T^N)} + \|\hat{\chi}\|_{L_1(T^N)} \\ &\le C\|\hat{\chi}\|_{W_1^k(T^N)} \\ &\le Ch^{-N}\|\chi\|_{W_1^k(\tau)}. \end{aligned} \tag{5.149}$$

Combining Equations 5.148 and 5.149, we see that

$$|E_\tau(\bar{f}\eta)| \le Ch^{q+2+k-r} \|\bar{f}\|_{W_\infty^{q+2+k-r}(\tau)} \|\chi\|_{W_1^k(\tau)}. \tag{5.150}$$

Equations 5.147 and 5.150 now give Equation 5.143.

Next we consider the cases where either $r = 3$ and $k = 2$, or $r = 2$ and $k = 1$. By Proposition 5.7,

$$|E_\tau(\bar{f}\chi)| \le Ch^{q+1+N} \|\bar{f}\|_{W_\infty^{q+1}(\tau)} \|\chi\|_{W_\infty^{q+1}(\tau)}. \tag{5.151}$$

By the element inverse property of S_h,

$$\begin{aligned} \|\chi\|_{W_\infty^{q+1}(\tau)} &\le C\|\chi\|_{W_\infty^{r-1}(\tau)} \\ &\le Ch^{-N}\|\chi\|_{W_1^{r-1}(\tau)}. \end{aligned} \tag{5.152}$$

Notice that $q + 1 = q + 2 + k - r$ and $r - 1 = k$. Therefore, combining Equations 5.151 and 5.152 gives Equation 5.143.

At this point we have shown Equation 5.143 except in the case where $r = 2$ and $k = 2$. Putting together Equations 5.18, 5.11, 5.126, and 5.143,

$$\begin{aligned} |(\lambda_h' - \lambda_h)(\chi)| &\le \sum_{\tau \in T_h} |E_\tau(\bar{f}\chi)| \\ &\le Ch^{q+2+k-r} \|\bar{f}\|_{W_\infty^{q+2+k-r}(\Omega_h)} \sum_{\tau \in T_h} \|\chi\|_{W_1^k(\tau)}. \end{aligned} \tag{5.153}$$

Together with Equation 5.32, the $k = 1$ case gives Equation 5.45 and the $k = 2$ case gives Equation 5.46.

5.6 Future Work

It would be nice to extend these results to handle the case of nonhomogeneous boundary conditions. To set this up, we let $\phi \in W_2^1(\Omega)$ and let $g : \partial\Omega \to \mathbb{R}$ be such that $g = \phi$ on $\partial\Omega$. Then, instead of requiring $u = 0$ on $\partial\Omega$, we require that $u = g$ on $\partial\Omega$.

Now let $\phi_h \in S_h$ be a good approximation of ϕ and let $g_h : \partial\Omega_h \to \mathbb{R}$ be such that $g_h = \phi_h$ on $\partial\Omega_h$. Then, instead of requiring $u_h = 0$ on $\partial\Omega_h$, we require that $u_h = g_h$ on $\partial\Omega_h$.

The naïve approach is to apply Theorem 5.3 to $(u - \phi) - (u_h - \phi_h) \circ \Phi_h^{-1}$, which makes sense because $u - \phi = 0$ on $\partial\Omega$ and $u_h - \phi_h = 0$ on $\partial\Omega_h$. Another approach would be to rework Theorem 5.3. An idea on how to proceed is given in [26, p. 420].

The only unsatisfactory aspect of our results is that the pattern in Theorem 5.1 for the L_∞ estimates is interrupted in the case $r = 2$. Here, no weight is possible. It appears that we can not get away with a quadrature rule that does not integrate linear functions exactly. This is consistent with a report in [30, p. 178] of the midpoint rule, which integrates linear functions exactly, being used to obtain optimal L_∞ estimates.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics, vol. 140, Academic Press, Oxford, 2003.

[2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions I*, Communications on Pure and Applied Mathematics **12** (1959), no. 4, 623–727.

[3] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, Lectures in Applied Mathematics (Proceedings of the Summer Seminar, Boulder, Colorado, 1957), vol. III, Interscience, New York, 1964.

[4] J. Bonet and R. D. Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis*, Cambridge University Press, Cambridge, 1997.

[5] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008.

[6] A. P. Calderon and A. Zygmund, *On Singular Integrals*, American Journal of Mathematics **78** (1956), no. 2, 289–309.

[7] P. G. Ciarlet, *Basic Error Estimates for Elliptic Problems*, Finite Element Methods (Part 1) (P. G. Ciarlet and J. L. Lions, eds.), Handbook of Numerical Analysis, vol. II, Elsevier, North Holland, 1991, pp. 17–352.

[8] P. G. Ciarlet and P. A. Raviart, *The Combined Effect of Curved Boundaries and Numerical Integration in Isoparametric Finite Element Method*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (New York) (A.K. Aziz, ed.), Academic Press, 1972, pp. 409–474.

[9] A. Demlow, *Sharply Localized Pointwise and W^{-1}_{∞} Estimates for Finite Element Methods for Quasilinear Problems*, Mathematics of Computation **76** (2007), no. 260, 1725–1741.

[10] L. C. Evans, *Partial Differential Equations*, 3rd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, 2002.

[11] A. Friedman, *Partial Differential Equations*, Krieger, Huntington, 1976.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[13] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[14] J. Guzmán, *Quadrature and Schatz's Pointwise Estimates in Finite Element Methods*, BIT Numerical Mathematics **45** (2005), no. 4, 695–707.

[15] W. Hoffman, A. H. Schatz, L. B. Wahblin, and G. Wittum, *Asymptotically Exact A Posteriori Estimators for the Pointwise Gradient Error on Each Element in Irregular Meshes. Part I: A Smooth Problem and Globally Quasi-Uniform Meshes*, Mathematics of Computation **70** (2001), no. 235, 897–909.

[16] F. John, *Plane Waves and Spherical Means Applied to Partial Differential Equations*, Dover, Mineola, 1955.

[17] J. P. Krasovskii, *Isolation of Singularities of the Green's Function*, Mathematics of the USSR: Izvestiya **1** (1967), no. 5, 935–966.

[18] M. Lenoir, *Optimal Isoparametric Finite Elements and Error Estimates for Domains Involving Curved Boundaries*, SIAM Journal on Numerical Analysis **23** (1986), no. 3, 562–580.

[19] A. H. Schatz, *Pointwise Error Estimates and Asymptotic Error Expansion Inequalities for the Finite Element Method on Irregular Grids, Part I: Global Estimates*, Mathematics of Computation **67** (1998), no. 223, 877–899.

[20] ______, *Pointwise Error Estimates and Asymptotic Error Expansion Inequalities for the Finite Element Method on Irregular Grids, Part II: Interior Estimates*, SIAM Journal on Numerical Analysis **38** (2000), no. 4, 1269–1293.

[21] ______, *Perturbations of Forms and Error Estimates for the Finite Element Method at a Point, with an Application to Improved Superconvergence Error Estimates for Subspaces that are Symmetric with Respect to a Point*, SIAM Journal on Numerical Analysis **42** (2005), no. 6, 2342–2365.

[22] ______, *Some New Local Error Estimates in Negative Norms with an Application to Local A Posteriori Error Estimation*, International Journal of Numerical Analysis and Modeling **3** (2006), no. 3, 371–376.

[23] A. H. Schatz and L. B. Wahlbin, *Interior Maximum-Norm Estimates for Fi-*

nite Element Methods, Mathematics of Computation **31** (1977), no. 138, 414–442.

[24] ———, *On the Quasi-optimality in L_∞ of the H_0^1 Projection into Finite Element Spaces*, Mathematics of Computation **38** (1982), no. 157, 1–21.

[25] ———, *Interior Maximum Norm Estimates for Finite Element Methods, Part II*, Mathematics of Computation **64** (1995), no. 211, 907–928.

[26] R. Scott, *Interpolated Boundary Conditions in the Finite Element Method*, SIAM Journal on Numerical Analysis **12** (1975), no. 3, 404–427.

[27] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[28] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Johann Ambrosius Barth, Heidelberg, 1995.

[29] L. B. Wahlbin, *Exact Constants in Certain L_p Estimates*, Unpublished.

[30] ———, *Maximum Norm Error Estimates in the Finite Element Method with Isoparametric Quadratic Elements and Numerical Integration*, RAIRO Numerical Analysis **12** (1978), no. 2, 173–202.

[31] ———, *Local Behavior in Finite Element Methods*, Finite Element Methods (Part 1) (P. G. Ciarlet and J. L. Lions, eds.), Handbook of Numerical Analysis, vol. II, Elsevier, North Holland, 1991, pp. 353–522.

[32] ———, *Lecture Notes for Math 728: Seminar in Partial Differential Equations*, Spring 1998, Cornell University.